Sheet 2

Due: To be handed in before 10.03.2023 at 12:00.

1. Exercise

The goal of this exercise is to show that for any n events $A_1, \dots, A_n$ it holds that

$$\mathbb{P}(A_1 \cup \dots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \dots < i_k \le n} \mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k}). \tag{1}$$

(a) Show that

$$\mathbb{1}_{A_1 \cup \dots \cup A_n} = 1 - \prod_{i=1}^{n} (1 - \mathbb{1}_{A_i}). \tag{2}$$

(b) Derive formula (1) from (2).

Hint: You can use the facts that $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$ for any event A and that the operator $\mathbb{E}$ is linear.

2. Exercise

Each of three people tosses a coin. We assume that all possible outcomes have the same probability. What is the probability of someone being the "odd one out"? That is, two of the players obtain the same outcome, while the "odd one out" gets a different outcome.

Hint: Start with writing down Ω and the event of being the "odd one out".

3. Exercise

An urn contains three red, three black and two white balls.

(a) Two balls are drawn from the urn without replacement. What is the probability that they are of different colors?

(b) Now, three balls are drawn from the urn without replacement. What is the probability that they are of exactly two different colors?

4. Exercise

The goal of this exercise is to compute the expectation of some well-known discrete random variables.

(a) Let X be the random variable taking values in the set $\{x_1, \dots, x_k\}$ with probability $p_1, \dots, p_k$. Compute $\mathbb{E}[X]$. What is $\mathbb{E}[X]$ in the special case $x_i = i$, $k = 6$ and $p_i = 1/6$ for $i \in \{1, \dots, 6\}$?

(b) Let X be a Binomial random variable with parameters n and $p \in (0, 1)$, that is $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0, \dots, n\}$. Compute $\mathbb{E}[X]$.

(c) Let X be a Geometric random variable with parameter $p \in (0, 1)$, that is $\mathbb{P}(X = k) = p(1-p)^k$, for $k \in \{0, 1, 2, \dots\}$. Compute $\mathbb{E}[X]$.

5. Exercise

(a) Let X be a random variable such that $X(\Omega) = \{1/k \colon k = 1, 2, \dots\}$ and $\mathbb{P}(X = 1/k) = 2^{-k}$. Compute $\mathbb{E}[X]$.

(b) Let X be a random variable such that $X(\Omega) = \{1/k \colon k = 1, 2, \dots\} \cup \{k \colon k = 2, 3, \dots\}$ and $\mathbb{P}(X = 1/k) = 2^{-(k+1)}$ for $k \ge 1$ and $\mathbb{P}(X = k) = 2^{-k}$ for $k \ge 2$. Compute $\mathbb{E}[X]$.