Sheet 2

Due: To be handed in before 10.03.2023 at 12:00.

1. Exercise

The goal of this exercise is to show that for any n events $A_1, \dots, A_n$ it holds that

$$\mathbb{P}(A_1 \cup \dots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \dots < i_k \le n} \mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k}). \tag{1}$$

(a) Show that

$$\mathbb{1}_{A_1 \cup \dots \cup A_n} = 1 - \prod_{i=1}^{n} (1 - \mathbb{1}_{A_i}). \tag{2}$$

(b) Derive formula (1) from (2).

Hint: You can use the facts that $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$ for any event A and that the operator $\mathbb{E}$ is linear.

Solution:

(a) For any set B, we have the identity $\mathbb{1}_B + \mathbb{1}_{B^c} = 1$. Also, for any two sets B_1 and B_2, we have the identity $\mathbb{1}_{B_1 \cap B_2} = \mathbb{1}_{B_1} \mathbb{1}_{B_2}$, which generalizes to $\mathbb{1}_{B_1 \cap \dots \cap B_n} = \prod_{i=1}^{n} \mathbb{1}_{B_i}$. Thus,

$$1 - \mathbb{1}_{A_1 \cup \dots \cup A_n} = \mathbb{1}_{(A_1 \cup \dots \cup A_n)^c} = \mathbb{1}_{A_1^c \cap \dots \cap A_n^c} = \prod_{i=1}^{n} \mathbb{1}_{A_i^c} = \prod_{i=1}^{n} (1 - \mathbb{1}_{A_i}),$$

from which we get

$$\mathbb{1}_{A_1 \cup \dots \cup A_n} = 1 - \prod_{i=1}^{n} (1 - \mathbb{1}_{A_i}).$$

(b) Using the hint, we can write that

$$\mathbb{P}(A_1 \cup \dots \cup A_n) = \mathbb{E}[\mathbb{1}_{A_1 \cup \dots \cup A_n}] = 1 - \mathbb{E}\left[\prod_{i=1}^{n} (1 - \mathbb{1}_{A_i})\right].$$

Now, for any given real numbers $b_1, \dots, b_n$, we have that

$$\prod_{i=1}^{n} (1 + b_i) = 1 + \sum_{k=1}^{n} \sum_{1 \le i_1 < \dots < i_k \le n} b_{i_1} \cdot \ldots \cdot b_{i_k}.$$

Hence, for any $\omega \in \Omega$,

$$\prod_{i=1}^{n} (1 - \mathbb{1}_{A_i}(\omega)) = 1 + \sum_{k=1}^{n} (-1)^k \sum_{1 \le i_1 < \dots < i_k \le n} \mathbb{1}_{A_{i_1}}(\omega) \cdot \ldots \cdot \mathbb{1}_{A_{i_k}}(\omega).$$

It follows that

$$-\prod_{i=1}^{n} (1 - \mathbb{1}_{A_i}) = -1 + \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \dots < i_k \le n} \mathbb{1}_{A_{i_1}} \cdot \ldots \cdot \mathbb{1}_{A_{i_k}}$$

and

$$\begin{aligned}\mathbb{P}(A_1 \cup \cdots \cup A_n) &= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbb{E}[\mathbb{1}_{A_{i_1}} \cdot \ldots \cdot \mathbb{1}_{A_{i_k}}] \\ &= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbb{E}[\mathbb{1}_{A_{i_1} \cap \cdots \cap A_{i_k}}] \\ &= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}).\end{aligned}$$

2. Exercise

Each of three people tosses a coin. We assume that all possible outcomes have the same probability. What is the probability of someone being the "odd one out"? That is, two of the players obtain the same outcome, while the "odd one out" gets a different outcome.

Hint: Start with writing down Ω and the event of being the "odd one out".

Solution:

In this experiment, there are eight possible outcomes depending on whether each toss results in H or T:

$$\Omega = \{HHH, THH, HTH, HHT, HTT, TTH, THT, TTT\}.$$

To compute the probability of being the "odd one out", it is more practical to consider the complement of this event:

$$\mathbb{P}(\text{being the "odd one out"}) = 1 - \mathbb{P}(HHH \text{ or } TTT) = 1 - \mathbb{P}(HHH) - \mathbb{P}(TTT) = 1 - \frac{2}{8} = \frac{3}{4}.$$

3. Exercise

An urn contains three red, three black and two white balls.

(a) Two balls are drawn from the urn without replacement. What is the probability that they are of different colors?

(b) Now, three balls are drawn from the urn without replacement. What is the probability that they are of exactly two different colors?

Solution:

(a) In the following, we use R, B and W to indicate if a ball is red, black or white so that the urn contains R_1, R_2, R_3, B_1, B_2, B_3, W_1 and W_2. If we ignore the order in which the balls are drawn, we have that

$$\begin{aligned}\Omega = \big\{ &\{W_1, W_2\}, \\ &\{W_1, R_1\}, \{W_1, R_2\}, \{W_1, R_3\}, \{W_2, R_1\}, \{W_2, R_2\}, \{W_2, R_3\}, \\ &\{W_1, B_1\}, \{W_1, B_2\}, \{W_1, B_3\}, \{W_2, B_1\}, \{W_2, B_2\}, \{W_2, B_3\}, \\ &\{R_1, R_2\}, \{R_1, R_3\}, \{R_2, R_3\}, \\ &\{R_1, B_1\}, \{R_1, B_2\}, \{R_1, B_3\}, \{R_2, B_1\}, \{R_2, B_2\}, \{R_2, B_3\}, \{R_3, B_1\}, \{R_3, B_2\}, \{R_3, B_3\}, \\ &\{B_1, B_2\}, \{B_1, B_3\}, \{B_2, B_3\} \big\}.\end{aligned}$$

Let $A = \{\text{The drawn balls are of different colors}\}$. Then

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{7}{1 + 6 \cdot 2 + 3 \cdot 2 + 9} = 1 - \frac{7}{28} = \frac{3}{4}.$$

Note that $|\Omega| = \binom{8}{2} = \frac{8!}{2!6!} = 28$. Here, we used the facts that all elements in Ω occur with the same probability and that

$$A^c = \big\{\{W_1, W_2\}, \{R_1, R_2\}, \{R_1, R_3\}, \{R_2, R_3\}, \{B_1, B_2\}, \{B_1, B_3\}, \{B_2, B_3\}\big\}.$$

(b) Now, the experiment is to draw three balls instead of two. There are (at least) two ways to compute the probability of $A = \{\text{The balls are of exactly two different colors}\}$. We can either compute $\mathbb{P}(A)$ directly or use $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$.

- Computing $\mathbb{P}(A)$ directly:

  Note that $\mathbb{P}(A) = |A|/|\Omega|$. It is not necessary to list all outcomes in Ω because we know that $|\Omega| = \binom{8}{3}$ (we again ignore the order in which the balls are drawn). To compute $|A|$, we need to count all possible configurations with $2W$ and R (call it A_1); $2W$ and B (call it A_2); $2R$ and B (call it A_3); $2B$ and R (call it A_4); $2R$ and W (call it A_5); $2B$ and W (call it A_6). We have $|A_1| = 3 = |A_2|$, $|A_3| = \binom{3}{2}\binom{3}{1} = 9 = |A_4|$ and $|A_5| = \binom{3}{2}\binom{2}{1} = 6 = |A_6|$. Hence, $|A| = 36$ and it follows that

$$\mathbb{P}(A) = \frac{36}{\frac{8!}{3!5!}} = \frac{36}{56} = \frac{9}{14}.$$

- Computing $\mathbb{P}(A^c)$ first:

  Let K be the event $K = \{\text{The balls are of the same color or of three different colors}\}$. The possible configurations for K are $3R$ or $3B$ (call the set of these K_1) or R, B and W (call it K_2). Then $|K_1| = 2$ and $|K_2| = \binom{3}{1}\binom{3}{1}\binom{2}{1} = 18$. Hence, $|K| = 20$ and

$$\mathbb{P}(A) = 1 - \frac{20}{56} = \frac{9}{14}.$$

4. Exercise

The goal of this exercise is to compute the expectation of some well-known discrete random variables.

(a) Let X be the random variable taking values in the set $\{x_1, \dots, x_k\}$ with probability $p_1, \dots, p_k$. Compute $\mathbb{E}[X]$. What is $\mathbb{E}[X]$ in the special case $x_i = i$, $k = 6$ and $p_i = 1/6$ for $i \in \{1, \dots, 6\}$?

(b) Let X be a <u>Binomial</u> random variable with parameters n and $p \in (0,1)$, that is $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0, \dots, n\}$. Compute $\mathbb{E}[X]$.

(c) Let X be a <u>Geometric</u> random variable with parameter $p \in (0,1)$, that is $\mathbb{P}(X = k) = p(1-p)^k$, for $k \in \{0, 1, 2, \dots\}$. Compute $\mathbb{E}[X]$.

Solution:

(a) By definition of the expectation, we have

$$\mathbb{E}[X] = \sum_{x \in X(\Omega)} x p(x) = \sum_{i=1}^{k} x_i p_i = \frac{1}{6} \sum_{i=1}^{6} i = \frac{7}{2}.$$

This is the average number when a fair die is thrown.

(b) The are at least two ways to compute the expectation of a Binomial random variable.

- 1^{st} method:

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

and, for $k \geq 1$,

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}.$$

Thus,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = n\sum_{k=1}^{n}\binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= n\sum_{k=0}^{n-1}\binom{n-1}{k} p^{k+1}(1-p)^{n-1-k} = np\sum_{k=0}^{n-1}\binom{n-1}{k} p^k (1-p)^{n-1-k} \\
&= np\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{p}{1-p}\right)^k (1-p)^{n-1} = np\left(1+\frac{p}{1-p}\right)^{n-1}(1-p)^{n-1} \\
&= np.
\end{aligned}$$

- 2^{nd} method:

  A Binomial random variable is by definition the sum of n (independent) Bernoulli random variables with parameter p, that is $X = X_1 + \ldots X_n$ with $X_1, \ldots, X_n \in \{0,1\}$ and $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p$ for $1 \leq i \leq n$, and $X_1, \ldots, X_n$ are independent (a notion which we will study in more detail). Then, $\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i]$ and $\mathbb{E}[X_i] = 1p + 0(1-p) = p$. Hence, $\mathbb{E}[X] = np$.

(c) A Geometric random variable is considered to model the waiting time for some specific event to occur (e.g. the arrival of a customer in a shop). Such event is known to arrive with probability p. If the waiting time is k, this means that the event did not occur in the first k trials before it occured in the next trial (e.g. no customer arrived in the first k minutes after the shop opened but in the $k+1^{st}$). This explains $\mathbb{P}(X = k) = p(1-p)^k$ (one success and k failures). We have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k\geq 0} kp(1-p)^k = p(1-p)\sum_{k=1}^{\infty} k(1-p)^{k-1} \\
&= p(1-p)\left[-\frac{d}{dp}\sum_{k=0}^{\infty}(1-p)^k\right] = p(1-p)\left[-\frac{d}{dp}\frac{1}{1-(1-p)}\right] \\
&= p(1-p)\frac{1}{p^2} = \frac{1-p}{p} = \frac{1}{p} - 1.
\end{aligned}$$

5. Exercise

(a) Let X be a random variable such that $X(\Omega) = \{1/k\colon k = 1, 2, \ldots\}$ and $\mathbb{P}(X = 1/k) = 2^{-k}$. Compute $\mathbb{E}[X]$.

(b) Let X be a random variable such that $X(\Omega) = \{1/k\colon k = 1, 2, \ldots\} \cup \{k\colon k = 2, 3, \ldots\}$ and $\mathbb{P}(X = 1/k) = 2^{-(k+1)}$ for $k \geq 1$ and $\mathbb{P}(X = k) = 2^{-k}$ for $k \geq 2$. Compute $\mathbb{E}[X]$.

Solution:

(a) Using the Taylor series expansion $\log(1-x) = -\sum_{k=1}^{\infty} \frac{1}{k} x^k$ for any $x \in (-1, 1)$,

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \frac{1}{k} 2^{-k} = -\log\left(1 - \frac{1}{2}\right) = \log(2).$$

(b) First, we check that the probabilities sum to 1.

$$\begin{aligned}\sum_{k=1}^{\infty}\mathbb{P}(X=1/k)+\sum_{k=2}^{\infty}\mathbb{P}(X=k) &= \sum_{k=1}^{\infty}2^{-(k+1)}+\sum_{k=2}^{\infty}2^{-k}=\sum_{k=2}^{\infty}2^{-k}+\sum_{k=2}^{\infty}2^{-k}\\ &= 2\sum_{k=2}^{\infty}2^{-k}=\sum_{k=2}^{\infty}2^{-(k-1)}=\sum_{k=1}^{\infty}2^{-k}=\frac{1}{1-\frac{1}{2}}-1=1.\end{aligned}$$

Then,

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^{\infty}\frac{1}{k}2^{-(k+1)}+\sum_{k=2}^{\infty}k2^{-k}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}2^{-k}+\frac{1}{2}\sum_{k=2}^{\infty}k2^{-(k-1)}\\ &= \frac{1}{2}\log(2)+\frac{1}{2}\left(\sum_{k=1}^{\infty}k2^{-(k-1)}-1\right)=\frac{1}{2}\log(2)+\frac{1}{2}\left(-\frac{d}{dx}\sum_{k=0}^{\infty}x^k\Big|_{x=1/2}-1\right)\\ &= \frac{1}{2}\log(2)+\frac{1}{2}\left(\frac{1}{(1-\frac{1}{2})^2}-1\right)=\frac{3+\log(2)}{2}.\end{aligned}$$