Sheet 3

Due: To be handed in before 17.03.2023 at 12:00.

1. Exercise

In a building with four floors, an elevator starts with three people at the ground floor.

(a) What is the probability that these people get off at exactly two different floors?

(b) Let X be the number of people who got off at the first floor. Compute $\mathbb{E}[X]$.

(c) Let X be as before and Y be the number of people who got off at the second floor. Compute $\mathbb{P}(X = 1|Y = 1)$.

2. Exercise

We have a box which contains three different coins. Each one of the coins has a different probability to show "H" (heads) after it is tossed. Call these probabilities p_j, $j \in \{1, 2, 3\}$. We are given $p_1 = 1/4$, $p_2 = 1/2$ and $p_3 = 3/4$.

(a) We select a coin from the box completely at random. When this coin is tossed, it shows "H". What is the conditional probability that the coin number j was the one selected?

(b) The same coin is tossed again. What is the conditional probability of obtaining "H" again?

Remark: The term "conditional" relates to the event {The coin shows "H" in the first toss}.

(c) Show the following result: Let $A_1, \dots, A_k$ be a partition of Ω and let B, C be two events such that $\mathbb{P}(B \cap C) > 0$ and $\mathbb{P}(A_i \cap B) > 0$ for every $i \in \{1, \dots, k\}$. Then,

$$\mathbb{P}(A_j|B \cap C) = \frac{\mathbb{P}(A_j|B)\mathbb{P}(C|A_j \cap B)}{\sum_{i=1}^{k} \mathbb{P}(A_i|B)\mathbb{P}(C|A_i \cap B)}.$$

(d) If the same coin shows "H" again at the second toss, what is the conditional probability that the coin number j was selected?

3. Exercise

Let X and Y be random variables such that X and Y are independent and for some $\lambda, \mu > 0$ we have $\mathbb{P}(X = k) = \frac{1}{k!}e^{-\lambda}\lambda^k$ and $\mathbb{P}(Y = k) = \frac{1}{k!}e^{-\mu}\mu^k$ for $k \in \{0, 1, 2, \dots\}$. (This means that X and Y have Possion distribution with rate λ and μ, respectively.)

(a) For $k \in \{0, 1, 2, \dots\}$, compute $\mathbb{P}(X + Y = k)$. Do you recognize the distribution of $X + Y$?

(b) Consider the event $\{X = i|X + Y = n\}$ for some fixed $n \in \{0, 1, 2, \dots\}$ and $i \in \{0, \dots, n\}$. Compute the probability of this event. What do you conclude about the conditional distribution of X given that $X + Y = n$?

(c) Deduce $\mathbb{E}[X|X + Y]$.

4. Exercise

Let X be a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathbb{E}[X^2] < \infty$. Also, let $\mathcal{B} = (B_i)_{i \in I}$ be a partition of Ω. By definition, we know that

$$\mathbb{E}[X|\mathcal{B}] = \sum_{\substack{i \in I: \\ \mathbb{P}(B_i) > 0}} \mathbb{E}[X|B_i]\mathbb{1}_{B_i}$$

with $\mathbb{E}[X|B_i] = \frac{\mathbb{E}[X \mathbb{1}_{B_i}]}{\mathbb{P}(B_i)}$ if $\mathbb{P}(B_i) > 0$. Show that

$$\mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{B}])^2\right] = \min_{\substack{(c_i)_{i\in I}: \\ \sum_{i\in I} c_i^2 \mathbb{P}(B_i) < \infty}} \mathbb{E}\left[\left(X - \sum_{i:\mathbb{P}(B_i)>0} c_i \mathbb{1}_{B_i}\right)^2\right].$$