Sheet 3

Due: To be handed in before 17.03.2023 at 12:00.

1. Exercise

In a building with four floors, an elevator starts with three people at the ground floor.

(a) What is the probability that these people get off at exactly two different floors?

(b) Let X be the number of people who got off at the first floor. Compute $\mathbb{E}[X]$.

(c) Let X be as before and Y be the number of people who got off at the second floor. Compute $\mathbb{P}(X = 1|Y = 1)$.

Solution:

(a) To compute the probability of the event $E = \{\text{The three people get off at exactly two different floors}\}$, we need to count first all the possibilities, that is $|\Omega|$. Here, we have $|\Omega| = 4^3 = 64$ because each of the three people has four possibilities (four possible floors to get off at). Since it seems reasonable (in the absence of any additional information) to assume that all possibilities have the same probability, we have that $\mathbb{P}(E) = |E|/|\Omega| = |E|/64$. To compute $|E|$, note first that there are $\binom{4}{2}$ pairs of floors at which people can get off ($\{1, 2\}, \{1, 3\}, \ldots, \{3, 4\}$). Now, fix a pair, for example $\{1, 2\}$. There are $2\binom{3}{1} = 2\binom{3}{2} = 6$ ways of splitting the three people in two groups such that each one of the groups gets off at floor 1 or 2. This means that $|E| = 6\binom{4}{2} = 6\frac{4!}{2!2!} = 36$. Thus, $\mathbb{P}(E) = 36/64 = 9/16$.

(b) To compute $\mathbb{E}[X]$, we first note that X takes values in the set $\{0, 1, 2, 3\}$. Hence,

$$\mathbb{E}[X] = \sum_{i=0}^{3} i\mathbb{P}(X = i) = \sum_{i=1}^{3} i\mathbb{P}(X = i).$$

We need to compute $\mathbb{P}(X = i)$ for $i \in \{1, 2, 3\}$. For $i = 1$, $|\{X = 1\}|$ is equal to 3 (the number of people) times the number of possibilities for the remaining two people to get off at other floors. Since each of the two remaining people has three other floors to get off at, we get $\mathbb{P}(X = 1) = 3 \cdot 3^2/|\Omega| = 27/64$. For $i = 2$, we count the number of groups of two out of three people that can get off at the first floor and the number of possibilities for the one remaining person. Thus, $|\{X = 2\}| = \binom{3}{2} \cdot 3 = 9$ and $\mathbb{P}(X = 2) = 9/|\Omega| = 9/64$. For $i = 3$, all three persons have to get off at the first floor, so $|\{X = 3\}| = 1$ and $\mathbb{P}(X = 3) = 1/64$.

<u>Remark:</u> Consistency check: $\mathbb{P}(X = 0) = \mathbb{P}(\text{all three persons get off at floor 2, 3 or 4}) = 3^3/|\Omega| = 27/64 = 1 - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) - \mathbb{P}(X = 3)$.

It follows that

$$\mathbb{E}[X] = \sum_{i=1}^{3} i\mathbb{P}(X = i) = \frac{1 \cdot 27 + 2 \cdot 9 + 3 \cdot 1}{64} = \frac{48}{64} = \frac{3}{4}.$$

<u>Remark:</u> Here is a very simple way of obtaining $\mathbb{E}[X] = 3/4$. Since the floors play similar roles, the (random) number of people getting off at some floor should have the same distribution for all floors. Hence, they should all have the same expectation. Thus, if we denote by Y, Z and W the number of people getting of the second, third, respectively fourth floor, then $\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[Z] = \mathbb{E}[W]$ and $4\mathbb{E}[X] = \mathbb{E}[X + Y + Z + W] = 3$, so $\mathbb{E}[X] = 3/4$.

(c) We have

$$\mathbb{P}(X = 1|Y = 1) = \frac{\mathbb{P}(X = 1, Y = 1)}{\mathbb{P}(Y = 1)}.$$

By the previous remark, $\mathbb{P}(Y = 1) = \mathbb{P}(X = 1) = 27/64$. To count $|\{X = 1, Y = 1\}|$, we note that there are $\binom{3}{2}$ different groups of two people A, B; then we multiply with 2 to account for the possibility that either A gets off at the first and B at the second floor or A gets off at the second and B at the first floor; lastly, we multiply with 2 to account for the two different possibilities where the third person gets off (third or forth floor). Thus, $|\{X=1,Y=1\}| = \binom{3}{2}\cdot 2\cdot 2 = 12$ and $\mathbb{P}(X=1|Y=1) = 12/27$.

2. Exercise

We have a box which contains three different coins. Each one of the coins has a different probability to show "H" (heads) after it is tossed. Call these probabilities p_j, $j\in\{1,2,3\}$. We are given $p_1 = 1/4$, $p_2 = 1/2$ and $p_3 = 3/4$.

(a) We select a coin from the box completely at random. When this coin is tossed, it shows "H". What is the conditional probability that the coin number j was the one selected?

(b) The same coin is tossed again. What is the conditional probability of obtaining "H" again?

Remark: The term "conditional" relates to the event {The coin shows "H" in the first toss}.

(c) Show the following result: Let $A_1,\dots,A_k$ be a partition of Ω and let B, C be two events such that $\mathbb{P}(B\cap C) > 0$ and $\mathbb{P}(A_i\cap B) > 0$ for every $i\in\{1,\dots,k\}$. Then,

$$\mathbb{P}(A_j|B\cap C) = \frac{\mathbb{P}(A_j|B)\mathbb{P}(C|A_j\cap B)}{\sum_{i=1}^k \mathbb{P}(A_i|B)\mathbb{P}(C|A_i\cap B)}.$$

(d) If the same coin shows "H" again at the second toss, what is the conditional probability that the coin number j was selected?

Solution:

Let us denote $A_j = \{\text{Coin number } j \text{ was selected}\}$ and $H_1 = \{\text{We obtain "}H\text{" at the first toss}\}$ and $H_2 = \{\text{We obtain "}H\text{" at the second toss}\}$.

(a) We have

$$\mathbb{P}(A_j|H_1) = \frac{\mathbb{P}(A_j\cap H_1)}{\mathbb{P}(H_1)} = \frac{\mathbb{P}(H_1|A_j)\mathbb{P}(A_j)}{\mathbb{P}(H_1)} = \frac{\mathbb{P}(H_1|A_j)\mathbb{P}(A_j)}{\sum_{i=1}^3 \mathbb{P}(H_1|A_i)\mathbb{P}(A_i)}.$$

With $\mathbb{P}(A_i) = 1/3$ for $i=1,2,3$ and $\mathbb{P}(H_1|A_j) = p_j$,

$$\mathbb{P}(A_j|H_1) = \frac{p_j\frac{1}{3}}{\sum_{i=1}^3 p_i\frac{1}{3}} = \frac{p_j}{\frac{1}{4}+\frac{1}{2}+\frac{3}{4}} = \frac{2}{3}p_j = \begin{cases} 1/6 & \text{if } j=1,\\ 1/3 & \text{if } j=2,\\ 1/2 & \text{if } j=3.\end{cases}$$

(b) We want to compute $\mathbb{P}(H_2|H_1)$.

$$\begin{aligned}\mathbb{P}(H_2|H_1) &= \frac{\mathbb{P}(H_2\cap H_1)}{\mathbb{P}(H_1)} = \frac{\sum_{i=1}^3 \mathbb{P}(H_1\cap H_2|A_i)\mathbb{P}(A_i)}{\sum_{i=1}^3 \mathbb{P}(H_1|A_i)\mathbb{P}(A_i)}\\ &= \frac{\sum_{i=1}^3 \mathbb{P}(H_1|A_i)\mathbb{P}(H_2|A_i)\frac{1}{3}}{\sum_{i=1}^3 \mathbb{P}(H_1|A_i)\frac{1}{3}} = \frac{\sum_{i=1}^3 p_i^2}{\sum_{i=1}^3 p_i} = \frac{\frac{1}{16}+\frac{1}{4}+\frac{9}{16}}{\frac{1}{4}+\frac{1}{2}+\frac{3}{4}} = \frac{14/16}{3/2} = \frac{7}{12},\end{aligned}$$

where from the first to the second line we used that the two coin tosses are independent of each other.

(c) We have that

$$\mathbb{P}(A_j|B\cap C) = \frac{\mathbb{P}(A_j\cap B\cap C)}{\mathbb{P}(B\cap C)}$$

with

$$\mathbb{P}(A_j\cap B\cap C) = \mathbb{P}(C|A_j\cap B)\mathbb{P}(A_j\cap B) = \mathbb{P}(C|A_j\cap B)\mathbb{P}(A_j|B)\mathbb{P}(B)$$

and

$$\mathbb{P}(B \cap C) = \sum_{i=1}^{k} \mathbb{P}(A_i \cap B \cap C) = \sum_{i=1}^{k} \mathbb{P}(C|A_i \cap B)\mathbb{P}(A_i \cap B) = \sum_{i=1}^{k} \mathbb{P}(C|A_i \cap B)\mathbb{P}(A_i|B)\mathbb{P}(B).$$

Note that $\mathbb{P}(B) \geq \mathbb{P}(B \cap C) > 0$. It follows that

$$\mathbb{P}(A_j|B \cap C) = \frac{\mathbb{P}(A_j|B)\mathbb{P}(C|A_j \cap B)}{\sum_{i=1}^{k} \mathbb{P}(A_i|B)\mathbb{P}(C|A_i \cap B)}.$$

(d) Here, we want to compute $\mathbb{P}(A_j|H_1 \cap H_2)$. Using the expression from question (c), we can write

$$\mathbb{P}(A_j|H_1 \cap H_2) = \frac{\mathbb{P}(A_j|H_1)\mathbb{P}(H_2|A_j \cap H_1)}{\sum_{i=1}^{3} \mathbb{P}(A_i|H_1)\mathbb{P}(H_2|A_i \cap H_1)},$$

where $\mathbb{P}(A_j|H_1) = \frac{2}{3}p_j$ was already calculated in question (a) and $\mathbb{P}(H_2|A_j \cap H_1) = \mathbb{P}(H_2|A_j) = p_j$, because the first coin toss being "H" does not alter the probability of getting "H" again in the second toss. Thus,

$$\mathbb{P}(A_j|H_1 \cap H_2) = \frac{\frac{2}{3}p_j^2}{\sum_{i=1}^{3} \frac{2}{3}p_i^2} = \frac{8}{7}p_j^2 = \begin{cases} 1/14 & \text{if } j = 1, \\ 2/7 & \text{if } j = 2, \\ 9/14 & \text{if } j = 3. \end{cases}$$

3. Exercise

Let X and Y be random variables such that X and Y are independent and for some $\lambda, \mu > 0$ we have $\mathbb{P}(X = k) = \frac{1}{k!}e^{-\lambda}\lambda^k$ and $\mathbb{P}(Y = k) = \frac{1}{k!}e^{-\mu}\mu^k$ for $k \in \{0, 1, 2, \dots\}$. (This means that X and Y have Possion distribution with rate λ and μ, respectively.)

(a) For $k \in \{0, 1, 2, \dots\}$, compute $\mathbb{P}(X + Y = k)$. Do you recognize the distribution of $X + Y$?

(b) Consider the event $\{X = i | X + Y = n\}$ for some fixed $n \in \{0, 1, 2, \dots\}$ and $i \in \{0, \dots, n\}$. Compute the probability of this event. What do you conclude about the conditional distribution of X given that $X + Y = n$?

(c) Deduce $\mathbb{E}[X|X + Y]$.

Solution:

(a) Let $k \in \{0, 1, 2, \dots\}$. We have that $\{X + Y = k\} = \bigcup_{i=0}^{k}\{X = i, Y = k - i\}$. Then $\mathbb{P}(X + Y = k) = \sum_{i=0}^{k} \mathbb{P}(X = i, Y = k - i)$ since $\{X = i, Y = k - i\} \cap \{X = j, Y = k - j\} = \emptyset$ for $i \neq j$. Using independence of X and Y, it follows that

$$\begin{aligned}\mathbb{P}(X + Y = k) &= \sum_{i=0}^{k} \mathbb{P}(X = i)\mathbb{P}(Y = k - i) = \sum_{i=0}^{k} \frac{e^{-\lambda}\lambda^i}{i!}\frac{e^{-\mu}\mu^{k-i}}{(k-i)!} \\ &= \frac{e^{-\lambda-\mu}}{k!}\sum_{i=0}^{k}\binom{k}{i}\lambda^i\mu^{k-i} = \frac{e^{-\lambda-\mu}}{k!}(\lambda + \mu)^k.\end{aligned}$$

This means that $X + Y$ has a Poisson distribution with parameter $\lambda + \mu$.

(b) We compute for $i = 0, \dots, n$

$$\begin{aligned}\mathbb{P}(X = i | X + Y = n) &= \frac{\mathbb{P}(X = i, X + Y = n)}{\mathbb{P}(X + Y = n)} = \frac{\mathbb{P}(X = i, Y = n - i)}{\mathbb{P}(X + Y = n)} \\ &= \frac{\mathbb{P}(X = i)\mathbb{P}(Y = n - i)}{\mathbb{P}(X + Y = n)} = \frac{e^{-\lambda}\lambda^i}{i!} \frac{e^{-\mu}\mu^{n-i}}{(n-i)!} \frac{n!}{e^{-\lambda-\mu}(\lambda+\mu)^n} \\ &= \frac{\lambda^i \mu^{n-i}}{(\lambda+\mu)^n} \frac{n!}{i!(n-i)!} = \binom{n}{i} \left(\frac{\lambda}{\lambda+\mu}\right)^i \left(1 - \frac{\lambda}{\lambda+\mu}\right)^{n-i}.\end{aligned}$$

This means that, conditionally on $\{X + Y = n\}$, the distribution of X is Binomial with parameters n and $\frac{\lambda}{\lambda+\mu}$.

(c) To compute $\mathbb{E}[X|X+Y]$, we use the fact that

$$\mathbb{E}[X|X+Y] = \sum_{n \geq 0} \mathbb{E}[X|X+Y=n] \mathbb{1}_{X+Y=n}.$$

According to (b), we have that $\mathbb{E}[X|X+Y=n] = n\frac{\lambda}{\lambda+\mu}$ (using the fact that the expectation of a Binomial random variable with parameters n and p is np). Thus,

$$\mathbb{E}[X|X+Y] = \sum_{n \geq 0} n \frac{\lambda}{\lambda+\mu} \mathbb{1}_{X+Y=n} = \frac{\lambda}{\lambda+\mu}(X+Y).$$

4. Exercise

Let X be a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathbb{E}[X^2] < \infty$. Also, let $\mathcal{B} = (B_i)_{i \in I}$ be a partition of Ω. By definition, we know that

$$\mathbb{E}[X|\mathcal{B}] = \sum_{\substack{i \in I: \\ \mathbb{P}(B_i) > 0}} \mathbb{E}[X|B_i] \mathbb{1}_{B_i}$$

with $\mathbb{E}[X|B_i] = \frac{\mathbb{E}[X\mathbb{1}_{B_i}]}{\mathbb{P}(B_i)}$ if $\mathbb{P}(B_i) > 0$. Show that

$$\mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{B}])^2\right] = \min_{\substack{(c_i)_{i \in I}: \\ \sum_{i \in I} c_i^2 \mathbb{P}(B_i) < \infty}} \mathbb{E}\left[\left(X - \sum_{i: \mathbb{P}(B_i) > 0} c_i \mathbb{1}_{B_i}\right)^2\right].$$

Solution:

Denote $\tilde{c}_i = \frac{\mathbb{E}[X\mathbb{1}_{B_i}]}{\mathbb{P}(B_i)}$ for $i \in I$ with $\mathbb{P}(B_i) > 0$ so that $\mathbb{E}[X|\mathcal{B}] = \sum_{i \in I: \mathbb{P}(B_i) > 0} \tilde{c}_i \mathbb{1}_{B_i}$. We claim that $\sum_{i: \mathbb{P}(B_i) > 0} \tilde{c}_i^2 \mathbb{P}(B_i) < \infty$. Indeed, we have $\mathbb{1}_{B_i} = \mathbb{1}_{B_i} \mathbb{1}_{B_i}$ and, hence, by the Cauchy-Schwarz inequality,

$$\tilde{c}_i^2 \mathbb{P}(B_i) = \frac{\mathbb{E}[X\mathbb{1}_{B_i}\mathbb{1}_{B_i}]^2}{\mathbb{P}(B_i)} \leq \frac{\mathbb{E}[X^2\mathbb{1}_{B_i}^2]\mathbb{E}[\mathbb{1}_{B_i}^2]}{\mathbb{P}(B_i)} = \frac{\mathbb{E}[X^2\mathbb{1}_{B_i}]\mathbb{E}[\mathbb{1}_{B_i}]}{\mathbb{P}(B_i)} = \mathbb{E}[X^2\mathbb{1}_{B_i}].$$

Thus,

$$\sum_{i: \mathbb{P}(B_i) > 0} \tilde{c}_i^2 \mathbb{P}(B_i) = \sum_{i: \mathbb{P}(B_i) > 0} \mathbb{E}[X^2 \mathbb{1}_{B_i}] = \mathbb{E}[X^2] < \infty.$$

Now, let $(c_i)_{i \in I}$ be any collection of real numbers such that $\sum_{i \in I} c_i^2 \mathbb{P}(B_i) < \infty$. For simplicity of notation, we assume from now on that $\mathbb{P}(B_i) > 0$ for all $i \in I$. We compute

$$\begin{aligned}\mathbb{E}\left[\left(X - \sum_{i\in I} c_i \mathbb{1}_{B_i}\right)^2\right] &= \mathbb{E}\left[\left(X - \mathbb{E}[X|\mathcal{B}] + \mathbb{E}[X|\mathcal{B}] - \sum_{i\in I} c_i \mathbb{1}_{B_i}\right)^2\right] \\ &= \mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{B}])^2\right] + 2\mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{B}])\left(\mathbb{E}[X|\mathcal{B}] - \sum_{i\in I} c_i \mathbb{1}_{B_i}\right)\right] \\ &\quad + \mathbb{E}\left[\left(\mathbb{E}[X|\mathcal{B}] - \sum_{i\in I} c_i \mathbb{1}_{B_i}\right)^2\right].\end{aligned}$$

Note that

$$\mathbb{E}[X|\mathcal{B}] - \sum_{i\in I} c_i \mathbb{1}_{B_i} = \sum_{i\in I} (\tilde{c}_i - c_i) \mathbb{1}_{B_i}.$$

Now, we show for any $(d_i)_{i\in I}$ with $\sum_{i\in I} d_i^2 \mathbb{P}(B_i) < \infty$ that

$$\mathbb{E}\left[(X - E[X|\mathcal{B}]) \sum_{i\in I} d_i \mathbb{1}_{B_i}\right] = 0.$$

We have

$$\mathbb{E}[X|\mathcal{B}] \sum_{i\in I} d_i \mathbb{1}_{B_i} = \left(\sum_{i\in I} \tilde{c} \mathbb{1}_{B_i}\right)\left(\sum_{i\in I} d_i \mathbb{1}_{B_i}\right) = \sum_{i,j\in I} \tilde{c}_i d_j \mathbb{1}_{B_i} \mathbb{1}_{B_j} = \sum_{i\in I} \tilde{c}_i d_i \mathbb{1}_{B_i}$$

since $\mathbb{1}_{B_i} \mathbb{1}_{B_j} = 0$ for $i \neq j$ and, hence,

$$\begin{aligned}\mathbb{E}\left[(X - E[X|\mathcal{B}]) \sum_{i\in I} d_i \mathbb{1}_{B_i}\right] &= \mathbb{E}\left[\sum_{i\in I} d_i X \mathbb{1}_{B_i} - \mathbb{E}[X|\mathcal{B}] \sum_{i\in I} d_i \mathbb{1}_{B_i}\right] \\ &= \sum_{i\in I} d_i \mathbb{E}[X \mathbb{1}_{B_i}] - \sum_{i\in I} \tilde{c}_i d_i \mathbb{P}(B_i) = 0\end{aligned}$$

since $\tilde{c}_i \mathbb{P}(B_i) = \mathbb{E}[X \mathbb{1}_{B_i}]$. With $d_i = \tilde{c}_i - c_i$, it follows that

$$\begin{aligned}\mathbb{E}\left[\left(X - \sum_{i\in I} c_i \mathbb{1}_{B_i}\right)^2\right] &= \mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{B}])^2\right] + \mathbb{E}\left[\left(\mathbb{E}[X|\mathcal{B}] - \sum_{i\in I} c_i \mathbb{1}_{B_i}\right)^2\right] \\ &\geq \mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{B}])^2\right],\end{aligned}$$

which shows that the minimum of the expectation of the quadratic error $(X - \sum c_i \mathbb{1}_{B_i})^2$ is attained for $c_i = \tilde{c}_i$, that is when $\sum c_i \mathbb{1}_{B_i} = \mathbb{E}[X|\mathcal{B}]$.