Sheet 4

Due: To be handed in before 24.03.2023 at 12:00.

1. Exercise

Let $(S_k)_{0\le k\le N}$ be a random walk with N steps for some integer $N \ge 1$. More precisely, $S_0 = 0$ and $S_k = \sum_{i=1}^k X_i$ for $1 \le k \le N$, where $(X_1, \ldots, X_N) \in \Omega = \{\omega = (x_1, \ldots, x_n) \colon x_i \in \{-1, 1\},\ 1 \le i \le N\} = \{-1, 1\}^N$, which is equipped with the (discrete) uniform distribution, i.e. $\mathbb{P}(\{\omega\}) = 2^{-N}$ for all $\omega \in \Omega$. For this exercise, we recall Stirling's formula for large n:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

(a) Write down $\mathbb{P}(S_{2n} = 0)$ and $\mathbb{P}(S_{2n-1} = 1)$ using the formula from the lecture or script. Show that these probabilities are equal.

(b) For n large, show that $\mathbb{P}(S_{2n} = 0) \sim 1/\sqrt{\pi n}$.

(c) Conclude that, for n large enough, $\mathbb{P}(S_n = 0) \sim 1/\sqrt{\pi n/2}$ if n is even and that the same holds for $\mathbb{P}(S_n = \pm 1)$ if n is odd.

2. Exercise

The goal of this question is to show, for $a > 0$, $b \ge -a$ and $1 \le n \le N$, that

$$\mathbb{P}(T_{-a} \le n,\ S_n = b) = \mathbb{P}(S_n = -2a - b),$$

where we recall that $T_c = \min\{k \in \{1, \ldots, N\} \colon S_k = c\}$ (with the convention $T_c = N + 1$ if the set is empty). For $\omega = (\omega_1, \ldots, \omega_N) \in \Omega = \{-1, 1\}^N$, we recall that the realization of a random walk with N steps for this ω is $(S_k(\omega))_{0\le k\le N}$, where $S_0 = 0$, $S_k(\omega) = \sum_{i=1}^k X_i(\omega)$ and $X_i(\omega) = \omega_i$. Consider the events $E_1 = \{\omega \in \Omega \colon T_{-a}(\omega) \le n, S_n(\omega) = b\}$ and $E_2 = \{\omega \in \Omega \colon S_n(\omega) = -2a - b\}$. Consider also the application $\phi \colon E_1 \to E_2$, $\omega \mapsto \omega' = \phi(\omega)$ defined as

$$\omega'_i = \begin{cases} \omega_i & \text{if } i \le T_{-a}(\omega), \\ -\omega_i & \text{if } i > T_{-a}(\omega). \end{cases}$$

(a) Show that we have indeed $\phi(E_1) \subseteq E_2$.

(b) Show that if $S_k = x$ for some $k \in \{1, \ldots, N\}$ and $x > 0$, then each value in $\{1, \ldots, x - 1\}$ must have been reached by the random walk before time k.

(c) Show that ϕ is a bijection from E_1 onto E_2.

(d) Conclude that $\mathbb{P}(T_{-a} \le n,\ S_n = b) = \mathbb{P}(S_n = -2a - b)$.

3. Exercise

In this exercise, we want to show the identity $\mathbb{P}(T_0 > 2n) = \mathbb{P}(S_{2n} = 0)$ $(\star)$. To this end, we will start by showing that $\mathbb{P}(T_0 \le 2n) = \mathbb{P}(T_{-1} \le 2n - 1)$.

(a) Show that T_0 is necessarily an even integer and T_{-1} is necessarily an odd integer.

(b) Consider the events

$$E_{-1} = \{\text{there is an even integer } k \in \{2, \ldots, 2n\} \text{ such that } S_{k-1} = -1 \text{ and } X_k = 1\},$$
$$E_{+1} = \{\text{there is an even integer } k \in \{2, \ldots, 2n\} \text{ such that } S_{k-1} = 1 \text{ and } X_k = -1\}$$

and show that $\mathbb{P}(T_0 \le 2n) = \mathbb{P}(E_{-1}) + \mathbb{P}(E_{+1}) - \mathbb{P}(E_{-1} \cap E_{+1}) = \mathbb{P}(T_{-1} \le 2n - 1)$.

Hint: Use symmetry of the distribution of a random walk.

(c) Using the result $\mathbb{P}(T_{-a} \le n) = \mathbb{P}(S_n \notin (-a, a])$ for $a > 0$, show the identity $(\star)$.

4. Exercise

(a) Use your favorite software to generate 100 (or more) independent random walks $(S_k)_{0\leq k\leq N}$ with $N = 500$.

Hint: It may be useful to note that a random variable X with $\mathbb{P}(X = \pm 1) = 1/2$ has the same distribution as 2Bernoulli(1/2)−1.

(b) Let (as in the lecture) $L = \max\{0 \leq k \leq 2N : S_k = 0\}$ denote the last time the random walk $(S_k)_{0\leq k\leq 2N}$ visited 0. Check the arcsin law, that is that the density of the random variable $Z_N = L/(2N) = L/1000$ is close to $z \mapsto 1/(\pi\sqrt{z(1-z)})$.

Hint: You can draw the histogram of the realizations of Z_N along with the plot of $z \mapsto 1/(\pi\sqrt{z(1-z)})$.