Sheet 4

Due: To be handed in before 24.03.2023 at 12:00.

1. Exercise

Let $(S_k)_{0\leq k\leq N}$ be a random walk with N steps for some integer $N \geq 1$. More precisely, $S_0 = 0$ and $S_k = \sum_{i=1}^k X_i$ for $1 \leq k \leq N$, where $(X_1, \ldots, X_N) \in \Omega = \{\omega = (x_1, \ldots, x_n) : x_i \in \{-1, 1\},\ 1 \leq i \leq N\} = \{-1, 1\}^N$, which is equipped with the (discrete) uniform distribution, i.e. $\mathbb{P}(\{\omega\}) = 2^{-N}$ for all $\omega \in \Omega$. For this exercise, we recall Stirling's formula for large n:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

(a) Write down $\mathbb{P}(S_{2n} = 0)$ and $\mathbb{P}(S_{2n-1} = 1)$ using the formula from the lecture or script. Show that these probabilities are equal.

(b) For n large, show that $\mathbb{P}(S_{2n} = 0) \sim 1/\sqrt{\pi n}$.

(c) Conclude that, for n large enough, $\mathbb{P}(S_n = 0) \sim 1/\sqrt{\pi n/2}$ if n is even and that the same holds for $\mathbb{P}(S_n = \pm 1)$ if n is odd.

Solution:

(a) We have, for any integers $n \geq 1$, $k \in \{0, \ldots, n\}$, that $\mathbb{P}(S_n = 2k - n) = \binom{n}{k} 2^{-n}$. Thus, $\mathbb{P}(S_{2n} = 0) = \binom{2n}{n} 2^{-2n}$ and $\mathbb{P}(S_{2n-1} = 1) = \binom{2n-1}{n} 2^{-(2n-1)}$. To show that these probabilities are equal, we note that

$$\binom{2n-1}{n} \frac{1}{2^{2n-1}} = \frac{(2n-1)!}{n!(n-1)!} \frac{2}{2^{2n}} = \frac{(2n)!}{n!n!} \frac{1}{2^{2n}} = \binom{2n}{n} \frac{1}{2^{2n}}.$$

(b) By Stirling's formula, for large n,

$$\frac{(2n)!}{(n!)^2} \sim \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{4\pi n}}{\left(\frac{n}{e}\right)^{2n} 2\pi n} = \frac{2^{2n}}{\sqrt{\pi n}},$$

which yields $\mathbb{P}(S_{2n} = 0) \sim 1/\sqrt{\pi n}$.

(c) Let $n = 2m$ with m large enough. Then,

$$\mathbb{P}(S_n = 0) = \mathbb{P}(S_{2m} = 0) \sim 1/\sqrt{\pi m} = 1/\sqrt{\pi n/2}.$$

If $n = 2m - 1$ with m large, then, by symmetry of the distribution of S_n,

$$\mathbb{P}(S_n = -1) = \mathbb{P}(S_n = 1) = \mathbb{P}(S_{2m-1} = 1) = \mathbb{P}(S_{2m} = 0) \sim 1/\sqrt{\pi m} \sim 1/\sqrt{\pi n/2},$$

where the latter apprximation holds for large n since $\lim_{n\to} \frac{n+1}{n} = 1$.

2. Exercise

The goal of this question is to show, for $a > 0$, $b \geq -a$ and $1 \leq n \leq N$, that

$$\mathbb{P}(T_{-a} \leq n,\ S_n = b) = \mathbb{P}(S_n = -2a - b),$$

where we recall that $T_c = \min\{k \in \{1, \ldots, N\} : S_k = c\}$ (with the convention $T_c = N + 1$ if the set is empty). For $\omega = (\omega_1, \ldots, \omega_N) \in \Omega = \{-1, 1\}^N$, we recall that the realization of a random walk with N steps for this ω is $(S_k(\omega))_{0\leq k\leq N}$, where $S_0 = 0$, $S_k(\omega) = \sum_{i=1}^k X_i(\omega)$ and $X_i(\omega) = \omega_i$. Consider the events

$E_1 = \{\omega \in \Omega\colon T_{-a}(\omega) \leq n,\, S_n(\omega) = b\}$ and $E_2 = \{\omega \in \Omega\colon S_n(\omega) = -2a - b\}$. Consider also the application $\phi\colon E_1 \to E_2$, $\omega \mapsto \omega' = \phi(\omega)$ defined as

$$\omega'_i = \begin{cases} \omega_i & \text{if } i \leq T_{-a}(\omega), \\ -\omega_i & \text{if } i > T_{-a}(\omega). \end{cases}$$

(a) Show that we have indeed $\phi(E_1) \subseteq E_2$.

(b) Show that if $S_k = x$ for some $k \in \{1, \ldots, N\}$ and $x > 0$, then each value in $\{1, \ldots, x-1\}$ must have been reached by the random walk before time k.

(c) Show that ϕ is a bijection from E_1 onto E_2.

(d) Conclude that $\mathbb{P}(T_{-a} \leq n,\ S_n = b) = \mathbb{P}(S_n = -2a - b)$.

Solution:

(a) It is clear that $\omega'_i \in \{-1, 1\}$ for any $\omega \in E_1$, $i \in \{1, \ldots, N\}$. We need to show that $S_n(\omega') = \sum_{i=1}^n \omega'_i = -2a-b$. We distinguish two cases. First, if $T_{-a}(\omega) = n$, then $\omega' = \omega$ by definition of ω' and $S_n(\omega) = -a$ by definition of T_{-a}. Since $\omega \in E_1$, this implies $S_n(\omega') = -a = S_n(\omega) = b$. Thus, this case can only occur when $b = -a$, in which case $S_n(\omega') = -a = -2a - b$. Hence, $\omega' \in E_2$. Secondly, if $T_{-a}(\omega) < n$, then

$$\begin{aligned} S_n(\omega') = \sum_{i=1}^n \omega'_i &= \sum_{i=1}^{T_{-a}(\omega)} \omega'_i + \sum_{i=T_{-a}(\omega)+1}^{n} \omega'_i = \sum_{i=1}^{T_{-a}(\omega)} \omega'_i - \sum_{i=T_{-a}(\omega)+1}^{n} \omega_i \\ &= \sum_{i=1}^{T_{-a}(\omega)} \omega'_i - \left(\sum_{i=1}^n \omega_i - \sum_{i=1}^{T_{-a}(\omega)} \omega_i \right) = 2 \sum_{i=1}^{T_{-a}(\omega)} \omega_i - \sum_{i=1}^n \omega_i \\ &= 2S_{T_{-a}(\omega)}(\omega) - S_n(\omega) = -2a - b. \end{aligned}$$

Hence, $\omega' \in E_2$ and we conclude that $\phi(E_1) \subseteq E_2$.

(b) We want to show that, for $x > 0$ and $n \in \{1, \ldots, N\}$, if $S_n = x$, then for all $y \in \{0, \ldots, x-1\}$ there exists $j \in \{0, \ldots, n-1\}$ such that $S_j = y$. We will show this using induction on n. For $n = 1$, we have $S_1 = x$ iff $X_1 = x$ iff $X_1 = 1 = x$ since $x > 0$. The property holds obviously in this case since $S_0 = 0$.

Suppose it is true for n and let us show it for $n+1$. Thus, suppose $S_{n+1} = x$. If $x = 1$, then there is nothing to show since $S_0 = 0$. Suppose that $x \geq 2$. We have either $S_n = x+1$ or $S_n = x-1$. Call these cases A and B. For A, we have by the inductive hypothesis on n that for all $y \in \{0, \ldots, x\}$ there exists $j \in \{0, \ldots, n-1\}$ such that $S_j = y$, implying that for all $y \in \{0, \ldots, x-1\}$ there exists $j \in \{0, \ldots, n\}$ such that $S_j = y$, and the property is true for $n+1$. For B, we have again by the inductive hypothesis on n that for all $y \in \{0, \ldots, x-2\}$ there exists $j \in \{0, \ldots, n-1\}$ such that $S_j = y$. This together with the fact that $S_n = x-1$ gives that for all $y \in \{0, \ldots, x-1\}$ there exists $j \in \{0, \ldots, n\}$ such that $S_j = y$, which completes the proof.

(c) We show that ϕ is injective: Let ω and $\tilde{\omega}$ such that $\phi(\omega) = \phi(\tilde{\omega}) = \omega'$. Then

$$\omega'_i = \left.\begin{cases} \omega_i & \text{if } i \leq T_{-a}(\omega), \\ -\omega_i & \text{if } i > T_{-a}(\omega) \end{cases}\right\} = \begin{cases} \tilde{\omega}_i & \text{if } i \leq T_{-a}(\tilde{\omega}), \\ -\tilde{\omega}_i & \text{if } i > T_{-a}(\tilde{\omega}). \end{cases}$$

Suppose that $T_{-a}(\omega) \neq T_{-a}(\tilde{\omega})$ and without loss of generality that $T_{-a}(\omega) < T_{-a}(\tilde{\omega})$. Then, for all $i \in \{1, \ldots, T_{-a}(\omega)\}$, $\omega'_i = \omega_i = \tilde{\omega}_i$. Hence,

$$\sum_{i=1}^{T_{-a}(\omega)} \tilde{\omega}_i = \sum_{i=1}^{T_{-a}(\omega)} \omega_i = S_{T_{-a}(\omega)}(\omega) = -a.$$

But this is impossible because $T_{-a}(\omega) < T_{-a}(\tilde{\omega})$ and $T_{-a}(\tilde{\omega})$ is the smallest integer $k \in \{1, \ldots, N\}$ such that $\sum_{i=1}^k \tilde{\omega} = -a$. Thus, we must have $T_{-a}(\omega) = T_{-a}(\tilde{\omega})$. This, in turn, implies that $\omega_i = \tilde{\omega}_i$

for all $1 \leq i \leq T_{-a}(\omega)$ and $-\omega_i = -\tilde{\omega}_i$ for all $i > T_{-a}(\omega)$. Hence, $\omega = \tilde{\omega}$ and ϕ is injective since ω and $\tilde{\omega}$ were chosen arbitrarily from E_1.

We show that ϕ is surjective: Let $\omega' \in E_2$. We will exhibit $\omega \in E_1$ such that $\omega' = \phi(\omega)$. We have that

$$S_n(\omega') = \sum_{i=1}^{n} \omega'_i = -2a - b = -a - (a+b) \leq -a$$

since $b \geq -a$. If $b = -a$, then $S_n(\omega') = -a$, so $T_{-a}(\omega') \leq n$ by definition of T_{-a}. If $b > -a$, then $S_n(\omega') \leq -a - 1$. By question (b) and a symmetry argument, we know that each value in $\{S_n(\omega')+1, S_n(\omega')+2, \dots, 0\}$ has been reached by the random walk $(S_k(\omega'))_{0 \leq k \leq N}$ at a point $< n$. Since $-a \in \{S_n(\omega')+1, \dots, 0\}$, we conclude that $T_{-a}(\omega') < n$ in this case. Hence, we always have that $T_{-a}(\omega') \leq n$. Let $\omega = (\omega_1, \dots, \omega_N)$ be the vector given by

$$\omega_i = \begin{cases} \omega'_i & \text{if } i \leq T_{-a}(\omega'), \\ -\omega'_i & \text{if } i > T_{-a}(\omega'). \end{cases}$$

Then, $\sum_{i=1}^{T_{-a}(\omega')} \omega_i = \sum_{i=1}^{T_{-a}(\omega')} \omega'_i = -a$ and, hence, $T_{-a}(\omega) \leq T_{-a}(\omega')$. If $T_{-a}(\omega) < T_{-a}(\omega')$, then we would have $\sum_{i=1}^{T_{-a}(\omega)} \omega'_i = -a$, which contradicts the definition of $T_{-a}(\omega')$. Thus, $T_{-a}(\omega) = T_{-a}(\omega')$. Hence, $T_{-a}(\omega) \leq n$.

We will now show that $S_n(\omega) = b$. If $n = T_{-a}(\omega)$, then

$$S_n(\omega) = \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{T_{-a}(\omega)} = -a$$

but also

$$S_n(\omega) = \sum_{i=1}^{n} \omega'_i = S_n(\omega') = -2a - b,$$

which means that $b = -a$. In this case, $S_n(\omega) = b$. If $n > T_{-a}(\omega)$, then

$$\begin{aligned} S_n(\omega) &= \sum_{i=1}^{T_{-a}(\omega)} \omega_i + \sum_{i=T_{-a}(\omega)+1}^{n} \omega_i = \sum_{i=1}^{T_{-a}(\omega)} \omega'_i - \sum_{i=T_{-a}(\omega)+1}^{n} \omega'_i \\ &= 2 \sum_{i=1}^{T_{-a}(\omega)} \omega'_i - \sum_{i=1}^{n} \omega'_i = -2a - (-2a - b) = b. \end{aligned}$$

It follows that $\omega \in E_1$ and it is clear that $\omega' = \phi(\omega)$. This concludes the proof of surjectivity.

(d) We have $\mathbb{P}(T_{-a} \leq n,\ S_n = b) = \mathbb{P}(E_1)$ and $\mathbb{P}(S_n = -2a - b) = \mathbb{P}(E_2)$. But $\mathbb{P}(E_1) = |E_1|/|\Omega|$ and $\mathbb{P}(E_2) = |E_2|/|\Omega|$, where $\mathbb{P}$ is the uniform probability measure on Ω. By question (c), we have that $|E_1| = |E_2|$ and the claimed equality follows.

3. Exercise

In this exercise, we want to show the identity $\mathbb{P}(T_0 > 2n) = \mathbb{P}(S_{2n} = 0)$ $(\star)$. To this end, we will start by showing that $\mathbb{P}(T_0 \leq 2n) = \mathbb{P}(T_{-1} \leq 2n - 1)$.

(a) Show that T_0 is necessarily an even integer and T_{-1} is necessarily an odd integer.

(b) Consider the events

$$\begin{aligned} E_{-1} &= \{\text{there is an even integer } k \in \{2, \dots, 2n\} \text{ such that } S_{k-1} = -1 \text{ and } X_k = 1\}, \\ E_{+1} &= \{\text{there is an even integer } k \in \{2, \dots, 2n\} \text{ such that } S_{k-1} = 1 \text{ and } X_k = -1\} \end{aligned}$$

and show that $\mathbb{P}(T_0 \leq 2n) = \mathbb{P}(E_{-1}) + \mathbb{P}(E_{+1}) - \mathbb{P}(E_{-1} \cap E_{+1}) = \mathbb{P}(T_{-1} \leq 2n - 1)$.

Hint: Use symmetry of the distribution of a random walk.

(c) Using the result $\mathbb{P}(T_{-a} \leq n) = \mathbb{P}(S_n \notin (-a, a])$ for $a > 0$, show the identity ($\star$).

Solution:

(a) Let $k = T_0 \in \mathbb{N}\backslash\{0\}$. Then $S_k = 0$. Also,

$$S_k = \sum_{i=1}^{k} X_i = \sum_{i=1}^{k} \mathbb{1}_{X_i=1} + \sum_{i=1}^{k}(-1)(1 - \mathbb{1}_{X_i=1}) = 2\sum_{i=1}^{k} \mathbb{1}_{X_i=1} - k.$$

Thus, $S_k = 0$ is equivalent to $2\sum_{i=1}^{k} \mathbb{1}_{X_i=1} = k$, which shows that $k = T_0$ has to be even.

Similarly, let $k = T_{-1} \in \mathbb{N}\backslash\{0\}$. Then $2\sum_{i=1}^{k} \mathbb{1}_{X_i=1} = k - 1$, so $T_{-1} - 1$ is even, i.e. T_{-1} is odd.

(b) Now, consider the event $E_0 = \{T_0 \leq 2n\}$. Then, by definition of T_0 and using (a), we have that

$$E_0 = \{\exists\, k \in \{2, 4, \dots, 2n\} \colon S_k = 0\} = E_{-1} \cup E_{+1}.$$

Then, $\mathbb{P}(E_0) = \mathbb{P}(E_{-1}) + \mathbb{P}(E_{+1}) - \mathbb{P}(E_{-1} \cap E_{+1})$. By symmetry, we have that

$$\mathbb{P}(E_{+1}) = \mathbb{P}(\exists\, k \in \{2, 4, \dots, 2n\} \colon S_{k-1} = -1 \text{ and } X_k = -1)$$

as well as

$$\mathbb{P}(E_{-1} \cap E_{+1}) = \mathbb{P}(\exists\, k_1 \neq k_2 \in \{2, 4, \dots, 2n\} \colon S_{k_1-1} = -1,\, X_{k_1} = 1,\, S_{k_2-1} = -1,\, X_{k_2} = -1).$$

Thus,

$$\begin{aligned}
\mathbb{P}(E_0) &= \mathbb{P}(\exists\, k \in \{2, 4, \dots, 2n\} \colon S_{k-1} = -1, X_k = 1) + \mathbb{P}(\exists\, k \in \{2, 4, \dots, 2n\} \colon S_{k-1} = -1, X_k = -1) \\
&\quad - \mathbb{P}(\exists\, k_1 \neq k_2 \in \{2, 4, \dots, 2n\} \colon S_{k_1-1} = -1,\, X_{k_1} = 1,\, S_{k_2-1} = -1,\, X_{k_2} = -1) \\
&= \mathbb{P}(\{\exists\, k \in \{2, 4, \dots, 2n\} \colon S_{k-1} = -1, X_k = 1\} \cup \{\exists\, k \in \{2, 4, \dots, 2n\} \colon S_{k-1} = -1, X_k = -1\}) \\
&= \mathbb{P}(\exists\, k \in \{2, 4, \dots, 2n\} \colon S_{k-1} = -1) = \mathbb{P}(\exists\, m \in \{1, 3, \dots, 2n-1\} \colon S_m = -1) \\
&= P(T_{-1} \leq 2n - 1).
\end{aligned}$$

(c) We have

$$\mathbb{P}(T_{-1} > 2n - 1) = \mathbb{P}(S_{2n-1} \in (-1, 1]) = \mathbb{P}(S_{2n-1} \in \{0, 1\}) = \mathbb{P}(S_{2n-1} = 1)$$

because $\mathbb{P}(S_{2n-1} = 0) = 0$. Using exercise 1(a), we conclude that $\mathbb{P}(T_0 > 2n) = \mathbb{P}(S_{2n} = 0)$.

4. Exercise

(a) Use your favorite software to generate 100 (or more) independent random walks $(S_k)_{0 \leq k \leq N}$ with $N = 500$.

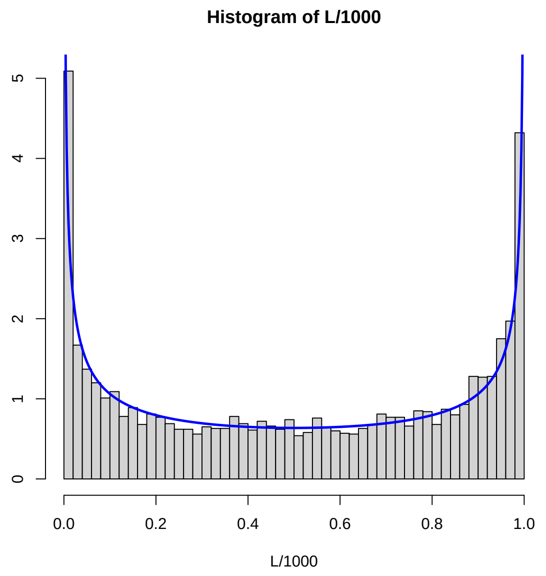
Hint: It may be useful to note that a random variable X with $\mathbb{P}(X = \pm 1) = 1/2$ has the same distribution as 2Bernoulli(1/2)−1.

(b) Let (as in the lecture) $L = \max\{0 \leq k \leq 2N \colon S_k = 0\}$ denote the last time the random walk $(S_k)_{0 \leq k \leq 2N}$ visited 0. Check the arcsin law, that is that the density of the random variable $Z_N = L/(2N) = L/1000$ is close to $z \mapsto 1/(\pi\sqrt{z(1-z)})$.

Hint: You can draw the histogram of the realizations of Z_N along with the plot of $z \mapsto 1/(\pi\sqrt{z(1-z)})$.

Solution:

See the code below for an example in the programming language R. The resulting histrogram and the plot of $z \mapsto 1/(\pi\sqrt{z(1-z)})$ are shown to the right.




```
###############A function which generates a random walk with 2N steps

SimFunRandWalk = function(N){

Bern.vec = rbinom(n = 2*N, size =1, prob=1/2)  #generate i.i.d. Bernoulli(1/2)
X.vec = 2*Bern.vec - 1   #generate i.i.d. Rademacher rvs (taking -1 or 1 with
    pb =1/2)

S = c(0,cumsum(X.vec))   # gives the random walk with 2N steps. Note that 0 is
    added
                          # at the beginning so that the length of S is 2N +1

return(S)
}

################## A function which determines the last time a vector "vec"
    takes the value 0
LFun  = function(vec){

index0 =  which(vec==0)  - 1  # substracting 1 is necessary because we vec[1]=0
    while this corresponds to S_0

L = max(index0)
return(L)
}


##################################Simulating M realizations from the same
    distribution of L######

Lout = NULL

M =5000
for(i in 1:M){
```

```
S.i = SimFunRandWalk(N=500)
Lout = c(Lout, LFun(S.i))
}


##############################    Computing Z = L/1000
Zout = Lout/(1000)

################################  The histogram of Z along with the density of the
     arsinus law

hist(Zout, nclass=40, prob=T, xlab="L/1000", ylab="", main="Histogram of L
    /1000")
points(sort(Zout), (1/pi)*1/(sqrt(sort(Zout)*(1-sort(Zout)))), col="blue", type
    ="l", lwd=2.5)
```