Sheet 5

Due: To be handed in before 07.04.2023 at 12:00.

1. Exercise

Consider the nonnegative function $f(x) = cx^{-k}\mathbb{1}_{x\geq 1}$ with $c > 0$.

(a) For what value of k is f a density function? Find the corresponding cdf and compute $\mathbb{P}(2 \leq X \leq 3)$, where $X \sim f$.

(b) Give an example of a density function f such that $c\sqrt{f}$ cannot be a density function for any $c \in (0, \infty)$.

2. Exercise

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$.

(a) Give the density of X and show that $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

(b) Let Φ be the cdf of $Z \sim \mathcal{N}(0, 1)$. Express the following probabilities in terms of μ, σ and Φ: $\mathbb{P}(X \leq 0)$, $\mathbb{P}(|X - \mu| \leq 2\sigma)$ and $\mathbb{P}(X > 3\mu)$.

(c) In this question, we assume that $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Z \sim \mathcal{N}(0, 1)$ are defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Now, we toss a coin that shows heads with probability $p \in (0, 1)$. We assume that the outcome of the toss is independent of X and Z. Define the random variable

$$Y = \begin{cases} X & \text{if the coin shows heads,} \\ Z & \text{if the coin shows tails.} \end{cases}$$

What are the cdf and pdf of Y? Do you know what such a distribution is called?

3. Exercise

(a) Let $X \sim \mathcal{U}([0, 1])$. Compute the cdf, the α-quantile for $\alpha \in (0, 1)$, $\mathbb{E}[X^n]$ and $\mathbb{E}[X^{1/n}]$ for all $n \geq 1$.

(b) Let $X \sim \mathrm{Beta}(\alpha, \beta)$ with $\alpha, \beta > 0$. This means that X is absolutely continuous with density

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1 - x)^{\beta-1}\mathbb{1}_{x\in(0,1)},$$

where $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$ for $a \in (0, \infty)$ is the Gamma function. Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

<u>Hint:</u> Note that $\int_0^1 x^{\alpha-1}(1 - x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ for any $\alpha, \beta > 0$ and that $\Gamma(a + 1) = a\Gamma(a)$ for any $a > 0$.

(c) Let $X \sim \mathrm{Exp}(\lambda)$ with $\lambda \in (0, \infty)$, i.e. X has density $f(x) = \lambda e^{-\lambda x}\mathbb{1}_{x>0}$. Compute the cdf, the α-quantile for $\alpha \in (0, 1)$ and $\mathbb{E}[X^n]$ for all $n \geq 1$.

<u>Hint:</u> use the normalizing constant in the density of a Gamma distribution.

(d) Let $X \sim \mathrm{Gamma}(\alpha, \beta)$ with $\alpha, \beta > 0$, i.e. X has density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\beta x}\mathbb{1}_{x>0}.$$

Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

4. Exercise

This exercise is mainly on uniform distributions.

(a) Suppose $X \sim \mathcal{U}([-\pi/2, \pi/2])$. Compute $\mathbb{E}[\sin(X)]$ and $\mathrm{Var}(\sin(X))$.

(b) (i) The lengths of the sides of a triangle are $2X$, $3X$ and $4X$ with $X \sim \mathcal{U}([0, \alpha])$ for some unknown $\alpha \in (0, \infty)$. Let A be the (random) area of the triangle. Find $\mathbb{E}[A]$ and $\mathrm{Var}(A)$.
Hint: You can use Heron's formula for the area of a triangle, that is $A = \sqrt{s(s-a)(s-b)(s-c)}$ with $s = (a+b+c)/2$ and a, b and c are the lengths of the sides of the triangle.

(ii) For what values of α is the probability that the area is bigger than 1 at least 50%?

(c) Let $X_1, \ldots, X_n$ be i.i.d. $\sim \mathcal{U}([0, 1])$. Put $I_n = \min_{1 \le i \le n} X_i$ and $M_n = \max_{1 \le i \le n} X_i$. Find the cdf of I_n and M_n, respectively. Can you recognize these distributions? Give $\mathbb{E}[I_n]$, $\mathrm{Var}(I_n)$, $\mathbb{E}[M_n]$ and $\mathrm{Var}(M_n)$.