Sheet 5

Due: To be handed in before 07.04.2023 at 12:00.

1. Exercise

Consider the nonnegative function $f(x) = cx^{-k}\mathbb{1}_{x\geq 1}$ with $c > 0$.

(a) For what value of k is f a density function? Find the corresponding cdf and compute $\mathbb{P}(2 \leq X \leq 3)$, where $X \sim f$.

(b) Give an example of a density function f such that $c\sqrt{f}$ cannot be a density function for any $c \in (0, \infty)$.

Solution:

(a) f is a density if and only if $x \mapsto x^{-k}\mathbb{1}_{x\geq 1}$ is integrable, that is $\int_1^\infty x^{-k}dx < \infty$. It is known that this is the case if and only if $k > 1$. In this case,

$$\int_1^\infty x^{-k}dx = \frac{1}{-k+1}x^{-k+1}\Big|_1^\infty = \frac{1}{k-1}$$

and, hence, $c = k - 1$ for $k \in (1, \infty)$. We have for all $x \in \mathbb{R}$ that

$$F(x) = \int_{-\infty}^x f(t)dt = \begin{cases} 0 & \text{if } x < 1, \\ \int_1^x \frac{k-1}{t^k}dt & \text{if } x \geq 1 \end{cases} = \begin{cases} 0 & \text{if } x < 1, \\ 1 - x^{-k+1} & \text{if } x \geq 1 \end{cases} = \left(1 - \frac{1}{x^{k-1}}\right)\mathbb{1}_{x\geq 1}.$$

Now, by continuity, $\mathbb{P}(X < 2) = \mathbb{P}(X \leq 2)$ and, hence,

$$\mathbb{P}(2 \leq X \leq 3) = \mathbb{P}(X \leq 3) - \mathbb{P}(X < 2) = F(3) - F(2) = \frac{1}{2^{k-1}} - \frac{1}{3^{k-1}}.$$

(b) Take $k = 2$. Then $f(x) = x^{-2}\mathbb{1}_{x\geq 1}$ is a density function but $x \mapsto cx^{-1}\mathbb{1}_{x\geq 1}$ is not for any $c \in (0, \infty)$.

2. Exercise

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$.

(a) Give the density of X and show that $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

(b) Let Φ be the cdf of $Z \sim \mathcal{N}(0, 1)$. Express the following probabilities in terms of μ, σ and Φ: $\mathbb{P}(X \leq 0)$, $\mathbb{P}(|X - \mu| \leq 2\sigma)$ and $\mathbb{P}(X > 3\mu)$.

(c) In this question, we assume that $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Z \sim \mathcal{N}(0, 1)$ are defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Now, we toss a coin that shows heads with probability $p \in (0, 1)$. We assume that the outcome of the toss is independent of X and Z. Define the random variable

$$Y = \begin{cases} X & \text{if the coin shows heads,} \\ Z & \text{if the coin shows tails.} \end{cases}$$

What are the cdf and pdf of Y? Do you know what such a distribution is called?

Solution:

(a) We know that $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if $X = \mu + \sigma Z$ with $Z \sim \mathcal{N}(0, 1)$. Then, $X = g(Z)$ with $g(z) = \mu + \sigma z$, $z \in \mathbb{R}$. Since g is bijective from $\mathbb{R}$ onto $\mathbb{R}$ and $g \in C^1(\mathbb{R})$ with $g'(z) = \sigma > 0$ for all

$z \in \mathbb{R}$, it follows that

$$f_X(x) = \frac{1}{\sigma} f_Z(g^{-1}(x)) = \frac{1}{\sigma} f_Z\Big(\frac{x-\mu}{\sigma}\Big),$$

where $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the density of Z. Thus,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

To show that $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$, we can use $X = \mu + \sigma Z$, which implies $\mathbb{E}[X] = \mu + \sigma\mathbb{E}[Z] = \mu$ since $\mathbb{E}[Z] = 0$. The latter holds because in

$$\mathbb{E}[Z] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z e^{-z^2/2} dz$$

we are integrating the odd function $ze^{-z^2/2}$ over the symmetric domain $\mathbb{R}$. Also, $\mathrm{Var}(X) = \mathrm{Var}(\mu + \sigma Z) = \sigma^2 \mathrm{Var}(Z)$ and

$$\mathrm{Var}(Z) = \mathbb{E}[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z^2 e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \left(-ze^{-z^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-z^2/2} dz \right) = 1.$$

(b) We have

$$\mathbb{P}(X \le 0) = \mathbb{P}(\mu + \sigma Z \le 0) = \mathbb{P}(Z \le -\mu/\sigma) = \Phi(-\mu/\sigma) = 1 - \Phi(\mu/\sigma)$$

and

$$\begin{aligned}\mathbb{P}(|X - \mu| \le 2\sigma) &= \mathbb{P}(|Z| \le 2) = \mathbb{P}(-2 \le Z \le 2) = \mathbb{P}(Z \le 2) - \mathbb{P}(Z < -2) \\ &= \Phi(2) - \Phi(-2) = \Phi(2) - (1 - \Phi(2)) = 2\Phi(2) - 1\end{aligned}$$

and

$$\mathbb{P}(X > 3\mu) = \mathbb{P}(X - \mu > 2\mu) = \mathbb{P}(Z > 2\mu/\sigma) = 1 - \mathbb{P}(Z \le 2\mu/\sigma) = 1 - \Phi(2\mu/\sigma).$$

(c) For any $y \in \mathbb{R}$, we have

$$\begin{aligned}F_Y(y) = \mathbb{P}(Y \le y) &= \mathbb{P}(Y \le y,\ \text{coin shows } H) + \mathbb{P}(Y \le y,\ \text{coin shows } T) \\ &= \mathbb{P}(X \le y,\ \text{coin shows } H) + \mathbb{P}(Z \le y,\ \text{coin shows } T) \\ &= \mathbb{P}(X \le y)p + \mathbb{P}(Z \le y)(1-p) \\ &= pF_X(y) + (1-p)\Phi(y) \\ &= p\Phi\Big(\frac{y-\mu}{\sigma}\Big) + (1-p)\Phi(y),\end{aligned}$$

where we used that $F_X(y) = \mathbb{P}(\mu + \sigma Z \le y) = \mathbb{P}(Z \le (y-\mu)/\sigma)$. Hence, the density of Y is

$$f_Y(y) = p\frac{1}{\sigma}\phi\Big(\frac{y-\mu}{\sigma}\Big) + (1-p)\phi(y),$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, $z \in \mathbb{R}$. We can view this as a mixture density with two components and mixing probability p.

3. Exercise

(a) Let $X \sim \mathcal{U}([0,1])$. Compute the cdf, the α-quantile for $\alpha \in (0,1)$, $\mathbb{E}[X^n]$ and $\mathbb{E}[X^{1/n}]$ for all $n \ge 1$.

(b) Let $X \sim \mathrm{Beta}(\alpha, \beta)$ with $\alpha, \beta > 0$. This means that X is absolutely continuous with density

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x \in (0,1)},$$

where $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$ for $a \in (0,\infty)$ is the Gamma function. Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Hint: Note that $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ for any $\alpha, \beta > 0$ and that $\Gamma(a+1) = a\Gamma(a)$ for any $a > 0$.

(c) Let $X \sim \mathrm{Exp}(\lambda)$ with $\lambda \in (0,\infty)$, i.e. X has density $f(x) = \lambda e^{-\lambda x}\mathbb{1}_{x>0}$. Compute the cdf, the α-quantile for $\alpha \in (0,1)$ and $\mathbb{E}[X^n]$ for all $n \geq 1$.

Hint: use the normalizing constant in the density of a Gamma distribution.

(d) Let $X \sim \mathrm{Gamma}(\alpha, \beta)$ with $\alpha, \beta > 0$, i.e. X has density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\beta x}\mathbb{1}_{x>0}.$$

Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Solution:

(a) X admits the density $f_X(x) = \mathbb{1}_{[0,1]}(x)$ and, hence, the cdf is

$$F_X(x) = \int_{-\infty}^x \mathbb{1}_{[0,1]}(x)dx = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

For $\alpha \in (0,1)$, we need to solve $F_X(x) = \alpha$. This gives that $x_\alpha = \alpha$ is the α-quantile. Lastly,

$$\mathbb{E}[X^n] = \int_0^1 x^n dx = \frac{1}{n+1},$$
$$\mathbb{E}[X^{1/n}] = \int_0^1 x^{1/n} dx = \frac{n}{n+1}.$$

(b) Using the hint, we compute

$$\mathbb{E}[X] = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^\alpha (1-x)^{\beta-1}dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)}$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{\alpha\Gamma(\alpha)}{(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta}.$$

To compute $\mathrm{Var}(X)$, we shall first compute $\mathbb{E}[X^2]$.

$$\mathbb{E}[X^2] = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha+1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+2+\beta)}$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{(\alpha+1)\alpha\Gamma(\alpha)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}.$$

Thus,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{\alpha}{\alpha+\beta}\left(\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right)$$
$$= \frac{\alpha}{\alpha+\beta} \frac{(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

(c) We have

$$F(x) = \int_{-\infty}^x f(t)dt = \int_{-\infty}^x \lambda e^{-\lambda t}\mathbb{1}_{t>0}dt = \begin{cases} 0 & \text{if } x < 0, \\ \int_0^x \lambda e^{-\lambda t}dt = 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$
$$= (1 - e^{-\lambda x})\mathbb{1}_{x \geq 0}.$$

Thus, for $x \geq 0$,

$$F(x) = \alpha \iff e^{-\lambda x} = 1 - \alpha \iff x = -\frac{\log(1-\alpha)}{\lambda},$$

so the α-quantile is $x_\alpha = \lambda^{-1}\log((1-\alpha)^{-1})$. Further,

$$\mathbb{E}[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} dx = \lambda \frac{\Gamma(n+1)}{\lambda^{n+1}} \int_0^\infty \underbrace{\frac{\lambda^{n+1}}{\Gamma(n+1)} x^{n+1-1} e^{-\lambda x}}_{\text{density of } \sim\Gamma(n+1,\lambda)} dx = \frac{n!}{\lambda^n}.$$

In particular, $\mathbb{E}[X] = 1/\lambda$ and $\mathrm{Var}(X) = 1/\lambda^2$.

(d) We have

$$\mathbb{E}[X] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1-1} e^{-\beta x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\alpha}{\beta}$$

and

$$\mathbb{E}[X^2] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+2-1} e^{-\beta x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} = \frac{\alpha(\alpha+1)}{\beta^2}$$

and, hence,

$$\mathrm{Var}(X) = \frac{\alpha(\alpha+1)}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}.$$

4. Exercise

This exercise is mainly on uniform distributions.

(a) Suppose $X \sim \mathcal{U}([-\pi/2, \pi/2])$. Compute $\mathbb{E}[\sin(X)]$ and $\mathrm{Var}(\sin(X))$.

(b) (i) The lengths of the sides of a triangle are $2X$, $3X$ and $4X$ with $X \sim \mathcal{U}([0,\alpha])$ for some unknown $\alpha \in (0,\infty)$. Let A be the (random) area of the triangle. Find $\mathbb{E}[A]$ and $\mathrm{Var}(A)$.
Hint: You can use Heron's formula for the area of a triangle, that is $A = \sqrt{s(s-a)(s-b)(s-c)}$ with $s = (a+b+c)/2$ and a, b and c are the lengths of the sides of the triangle.

(ii) For what values of α is the probability that the area is bigger than 1 at least 50%?

(c) Let $X_1, \ldots, X_n$ be i.i.d. $\sim \mathcal{U}([0,1])$. Put $I_n = \min_{1\leq i\leq n} X_i$ and $M_n = \max_{1\leq i\leq n} X_i$. Find the cdf of I_n and M_n, respectively. Can you recognize these distributions? Give $\mathbb{E}[I_n]$, $\mathrm{Var}(I_n)$, $\mathbb{E}[M_n]$ and $\mathrm{Var}(M_n)$.

Solution:

(a) We have

$$\mathbb{E}[\sin(X)] = \int_{-\pi/2}^{\pi/2} \frac{1}{\pi} \sin(x) dx = 0$$

because $\sin(x)$ is an odd function. Further,

$$\mathrm{Var}(\sin(X)) = \mathbb{E}[\sin(X)^2] - \mathbb{E}[\sin(X)]^2 = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin(x)^2 dx.$$

Recall that $\cos(2x) = \cos(x)^2 - \sin(x)^2 = 1 - 2\sin(x)^2$. Thus, $\sin(x)^2 = (1-\cos(2x))/2$ and, hence,

$$\mathrm{Var}(\sin(X)) = \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos(2x) dx\right) = \frac{1}{2} - \frac{1}{\pi}\frac{1}{4} \sin(2x)\Big|_{-\pi/2}^{\pi/2} = \frac{1}{2}.$$

(b) (i) We have $s = 9X/2$ and

$$A = \sqrt{\frac{9X}{2}\Big(\frac{9X}{2} - 2X\Big)\Big(\frac{9X}{2} - 3X\Big)\Big(\frac{9X}{2} - 4X\Big)} = \frac{3\sqrt{15}}{4}X^2.$$

Thus,

$$\mathbb{E}[A] = \frac{3\sqrt{15}}{4}\frac{1}{\alpha}\int_0^\alpha x^2 dx = \frac{\sqrt{15}}{4}\alpha^2$$

and

$$\mathbb{E}[A^2] = \left(\frac{3\sqrt{15}}{4}\right)^2 \frac{1}{\alpha}\int_0^\alpha x^4 dx = \frac{27}{16}\alpha^4.$$

Hence,

$$\mathrm{Var}(A) = \frac{27}{16}\alpha^4 - \left(\frac{\sqrt{15}}{4}\alpha^2\right)^2 = \frac{27}{16}\alpha^4 - \frac{15}{16}\alpha^4 = \frac{3}{4}\alpha^4.$$

(ii) We compute

$$\begin{aligned}\mathbb{P}(A \geq 1) = \mathbb{P}\left(\frac{3\sqrt{15}}{4}X^2 \geq 1\right) = \mathbb{P}\left(X^2 \geq \frac{4}{3\sqrt{15}}\right) &= 1 - \mathbb{P}\left(X \leq \sqrt{\frac{4}{3\sqrt{15}}}\right) \\ &= 1 - F_X\left(\sqrt{\frac{4}{3\sqrt{15}}}\right).\end{aligned}$$

Thus, $\mathbb{P}(A \geq 1) \geq 1/2$ if and only if $F_X\left(\sqrt{\frac{4}{3\sqrt{15}}}\right) \leq 1/2$. On the other hand,

$$F_X(x) = \int_{-\infty}^x \frac{1}{\alpha}\mathbb{1}_{[0,\alpha]}(x)dx = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x}{\alpha} & \text{if } 0 \leq x < \alpha, \\ 1 & \text{if } x \geq \alpha. \end{cases}$$

Hence, $F_X(x) \leq 1/2$ iff $x/\alpha \leq 1/2$ iff $\alpha \geq 2x$. We conclude that α must be $\geq \frac{4}{\sqrt{3\sqrt{15}}} \approx 1.173$.

(c) We compute

$$\begin{aligned}\mathbb{P}(I_n \leq x) &= 1 - \mathbb{P}(I_n > x) = 1 - \mathbb{P}\big(\min_{1\leq i\leq n} X_i > x\big) \\ &= 1 - \mathbb{P}(X_1 > x, \ldots, X_n > x) \overset{\text{independence}}{=} 1 - \prod_{i=1}^n \mathbb{P}(X_i > x) \\ &= 1 - \mathbb{P}(X_1 > x)^n\end{aligned}$$

since $X_1, \ldots, X_n$ all have the same distribution. Thus,

$$\mathbb{P}(I_n \leq x) = 1 - (1 - F_{X_1}(x))^n = \begin{cases} 0 & \text{if } x < 0, \\ 1 - (1-x)^n & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Similarly,

$$\mathbb{P}(M_n \leq x) = \prod_{i=1}^n \mathbb{P}(X_i \leq x) = F_{X_1}(x)^n = \begin{cases} 0 & \text{if } x < 0, \\ x^n & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

It follows that I_n admits the density $x \mapsto n(1-x)^{n-1}\mathbb{1}_{(0,1)}(x)$ and M_n the density $x \mapsto nx^{n-1}\mathbb{1}_{(0,1)}(x)$. Hence, $I_n \sim \mathrm{Beta}(1,n)$ and $M_n \sim \mathrm{Beta}(n,1)$. Using question 3(b), we find $\mathbb{E}[I_n] = \frac{1}{n+1}$, $\mathrm{Var}(I_n) = \frac{n}{(n+1)^2(n+2)}$ and $\mathbb{E}[M_n] = \frac{n}{n+1}$, $\mathrm{Var}(M_n) = \frac{n}{(n+1)^2(n+2)}$.