Sheet 6

Due: To be handed in before 07.04.2023 at 12:00.

1. Exercise

Consider the following joint probability density of some random pair (X, Y) for some $c > 0$;

$$f(x,y) = \begin{cases} cxy & \text{if } 1 \leq x \leq 3,\ 1 \leq y \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c.

(b) Are the random variables X and Y independent?

(c) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $\mathbb{E}[XY]$.

Solution:

(a) To be a density, f must satisfy

$$1 = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\, dxdy = c \int_1^3 \int_1^3 xy\, dxdy = c \left(\int_1^3 xdx \right)^2 = 16c.$$

Thus, $c = 1/16$.

(b) We note that

$$f(x,y) = \frac{1}{16} xy \mathbb{1}_{[1,3]}(x) \mathbb{1}_{[1,3]}(y) = f_1(x) f_2(y)$$

with $f_1(x) = \frac{1}{4} x \mathbb{1}_{[1,3]}(x)$ and $f_2(y) = \frac{1}{4} y \mathbb{1}_{[1,3]}(y)$ the respective pdfs of X and Y. Hence, X and Y are independent.

(c) Since X and Y have the same marginal density, we have that

$$\mathbb{E}[X] = \mathbb{E}[Y] = \int_{\mathbb{R}} x f_1(x) dx = \frac{1}{4} \int_1^3 x^2 dx = \frac{13}{6}.$$

Since X and Y are independent and, in particular, uncorrelated, we know that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = \frac{169}{36}$.

2. Exercise

Let f be the following joint probability density of some random pair (X, Y) for some $\alpha > 0$;

$$f(x,y) = \alpha \frac{1}{x^2} \mathbb{1}_{1 \leq x \leq y} \mathbb{1}_{[1,2]}(y).$$

(a) Find α.

(b) Are the random variables X and Y independent?

(c) Compute the covariance $\mathrm{cov}(X, Y)$ and the correlation $\rho(X, Y)$.

Solution:

(a) To be density, f must satisfy

$$1 = \iint_{\mathbb{R}^2} f(x,y)\,dxdy = \alpha \int_1^2 \int_1^y \frac{1}{x^2}\,dxdy = \alpha \int_1^2 (1 - y^{-1})dy = \alpha(1 - \log 2).$$

Hence, $\alpha = 1/(1 - \log 2)$.

(b) Using that $\mathbb{1}_{1 \le x \le y} \mathbb{1}_{[1,2]}(y) = \mathbb{1}_{[x,2]}(y)\mathbb{1}_{[1,2]}(x)$, the marginal density of X is given by

$$f_1(x) = \int_{\mathbb{R}} f(x,y)\,dy = \alpha \int_{\mathbb{R}} \frac{1}{x^2} \mathbb{1}_{[x,2]}(y)\mathbb{1}_{[1,2]}(x)\,dy = \alpha \frac{1}{x^2} \mathbb{1}_{[1,2]}(x) \int_x^2 dy = \alpha \frac{2-x}{x^2} \mathbb{1}_{[1,2]}(x).$$

Also,

$$f_2(y) = \alpha \int_{\mathbb{R}} \frac{1}{x^2} \mathbb{1}_{1 \le x \le y} \mathbb{1}_{[1,2]}(y)\,dx = \alpha \mathbb{1}_{[1,2]}(y) \int_1^y \frac{1}{x^2}\,dx = \alpha \frac{y-1}{y} \mathbb{1}_{[1,2]}(y).$$

It is clear that $f(x,y) \neq f_1(x) f_2(y)$ for some $(x,y) \in \mathbb{R}^2$ and, therefore, X and Y are not independent.

(c) By definition, $\mathrm{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. We have

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f_1(x)\,dx = \alpha \int_1^2 \frac{x(2-x)}{x^2} dx = \alpha \left(\int_1^2 \frac{2}{x}\,dx - \int_1^2 dx \right) = \alpha(2\log 2 - 1)$$

and

$$\mathbb{E}[Y] = \int_{\mathbb{R}} y f_2(y)\,dy = \alpha \int_1^2 y(1 - y^{-1})\,dy = \alpha \left(\frac{3}{2} - 1 \right) = \frac{\alpha}{2}$$

and

$$\begin{aligned}
\mathbb{E}[XY] &= \iint_{\mathbb{R}^2} xy f(x,y)\,dxdy = \alpha \int_1^2 \int_1^y \frac{xy}{x^2}\,dxdy \\
&= \alpha \int_1^2 y \int_1^y \frac{1}{x}\,dxdy = \alpha \int_1^2 y \log y\,dy \overset{\text{integration}}{\underset{\text{by parts}}{=}} \alpha \left(\frac{y^2}{2} \log y \Big|_1^2 - \frac{1}{2} \int_1^2 y^2 \frac{1}{y}\,dy \right) \\
&= \alpha \Big(2\log 2 - \frac{3}{4} \Big).
\end{aligned}$$

Plugging in $\alpha = 1/(1 - \log 2)$, it follows that

$$\begin{aligned}
\mathrm{cov}(X,Y) &= \alpha \Big(2\log 2 - \frac{3}{4} \Big) - \frac{\alpha^2}{2}(2\log 2 - 1) = \frac{\alpha^2}{2} \Big(\frac{4\log 2 - \frac{3}{2}}{\alpha} - 2\log 2 + 1 \Big) \\
&= \frac{1}{2(1-\log 2)^2} \Big((1 - \log 2)\big(4\log 2 - \frac{3}{2}\big) - 2\log 2 + 1 \Big) \\
&= \frac{1}{2(1-\log 2)^2} \Big(4\log 2(1 - \log 2) - \frac{\log 2}{2} - \frac{1}{2} \Big).
\end{aligned}$$

By definition,

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}.$$

We compute

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f_1(x)\,dx = \alpha \int_1^2 \frac{x^2(2-x)}{x^2} dx = \alpha \int_1^2 (2-x) dx = \frac{\alpha}{2}$$

and, hence,

$$\begin{aligned}
\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{\alpha}{2} - \alpha^2(2\log 2 - 1)^2 &= \frac{\alpha^2}{2} \Big(\frac{1}{\alpha} - 2(2\log 2 - 1)^2 \Big) \\
&= \frac{1}{2(1-\log 2)^2} \Big(1 - \log 2 - 2(2\log 2 - 1)^2 \Big).
\end{aligned}$$

Similarly,

$$\mathbb{E}[Y^2] = \int_{\mathbb{R}} y^2 f_2(y)\, dy = \alpha \int_1^2 y^2 (1 - y^{-1})\, dy = \frac{5\alpha}{6}$$

and

$$\begin{aligned}\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \frac{5\alpha}{6} - \frac{\alpha^2}{4} = \frac{\alpha^2}{2}\Big(\frac{5}{3\alpha} - \frac{1}{2}\Big) &= \frac{1}{2(1-\log 2)^2}\Big(\frac{5(1-\log 2)}{3} - \frac{1}{2}\Big) \\ &= \frac{1}{2(1-\log 2)^2}\Big(\frac{7}{6} - \frac{5\log 2}{3}\Big).\end{aligned}$$

Thus,

$$\rho(X,Y) = \sqrt{6}\frac{4\log 2(1-\log 2) - \frac{\log 2}{2} - \frac{1}{2}}{\sqrt{\big(1 - \log 2 - 2(2\log 2 - 1)^2\big)(7 - 10\log 2)}} \approx 0.429.$$

3. Exercise

Let X and Y be two i.i.d. $\sim \mathcal{G}(p)$ geometric random variables for some $p \in (0,1)$. Compute $\mathbb{P}(X \geq Y)$ and $\mathbb{P}(X > Y)$.

Solution:

Since X and Y play similar roles, we have that $\mathbb{P}(X \geq Y) = \mathbb{P}(Y \geq X)$. On the other hand,

$$\begin{aligned}\mathbb{P}(X \geq Y) + \mathbb{P}(X < Y) = 1 &\iff \mathbb{P}(X \geq Y) + \mathbb{P}(X \leq Y) - \mathbb{P}(X = Y) = 1 \\ &\iff 2\mathbb{P}(X \geq Y) = 1 + \mathbb{P}(X = Y) \\ &\iff \mathbb{P}(X \geq Y) = \frac{1 + \mathbb{P}(X = Y)}{2}.\end{aligned}$$

If f_{joint} denotes the joint probability mass function of (X, Y), then $\mathbb{P}(X = Y) = \sum_{x,y\geq 0} \mathbb{1}_{x=y} f_{\text{joint}}(x,y)$. Since X and Y are independent, we have that $f_{\text{joint}}(x,y) = f(x)f(y)$, where $f(x) = p(1-p)^x$ for $x = 0, 1, 2, \dots$. Thus,

$$\mathbb{P}(X = Y) = \sum_{x\geq 0} f(x)^2 = p^2 \sum_{x \geq 0} (1-p)^{2x} = \frac{p^2}{1 - (1-p)^2} = \frac{p}{2-p}.$$

We conclude that

$$\mathbb{P}(X \geq Y) = \frac{1}{2}\Big(1 + \frac{p}{2-p}\Big) = \frac{1}{2-p}$$

and

$$\mathbb{P}(X > Y) = \mathbb{P}(X \geq Y) - \mathbb{P}(X = Y) = \frac{1-p}{2-p}.$$