Sheet 7

Due: To be handed in before 21.04.2023 at 12:00.

1. Exercise

Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $X_n(\omega) \nearrow$ for $\forall \omega \in \Omega$ and $X_n(\omega) \geq 0$. Set $X_\infty(\omega) = \lim_{n\to\infty} X_n(\omega)$.

The Beppo Levi's Theorem says that

$$\mathbb{E}(X_\infty) = \lim_{n\to\infty} \mathbb{E}(X_n).$$

Use this result to show that if X is a random variable, then

$$\mathbb{E}(|X|) = 0 \Leftrightarrow \mathbb{P}(X = 0) = 1.$$

Hint:

- Consider $Y_n = |X|\mathbb{1}_{\{|X|\leq n\}}$ for "$\Leftarrow$".
- Write $\{X = 0\} = \bigcap_{n=1}^{\infty} \{|X| \leq \frac{1}{n}\}$ for "$\Rightarrow$".

2. Exercise

(a) Let $(X_n)_{n\geq 1}$ be a sequence of random variables and X be a random variable, all defined on the same probability space. We write that $X_n \xrightarrow{r} X$ or $X_n \xrightarrow{L_r} X$ for $r > 0$ if $\lim_{n\to\infty} \mathbb{E}[|X - X_n|^r] = 0$. Show that $X_n \xrightarrow{r} X$ implies $X_n \xrightarrow{\mathbb{P}} X$.

(b) Give an example of a sequence $(X_n)_{n\geq 1}$ such that $X_n \xrightarrow{\mathbb{P}} 0$ but not $X_n \xrightarrow{L_2} 0$.

(c) Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $\mathbb{P}(X_n = 0) = 1 - n^{-\alpha}$ and $\mathbb{P}(X_n = \sqrt{n}) = n^{-\alpha}$ for all $n \geq 1$ and some $\alpha > 0$. Show that if $\alpha > 1$, then $X_n \to 0$ a.s.

(d) Consider $Z \sim \mathcal{U}([0,1])$ and the random sequence $(X_n)_{n\geq 1}$ defined as $X_n = \mathbb{1}_{Z\in[m2^{-k},(m+1)2^{-k})}$ if $n = 2^k + m$ with $m \in \{0, 1, \dots, 2^k - 1\}$ and $k \in \{0, 1, \dots\}$. Show that $X_n \xrightarrow{\mathbb{P}} 0$ but that $X_n \overset{\text{a.s.}}{\nrightarrow} 0$.

3. Exercise

Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $X_n \xrightarrow{\mathbb{P}} c$ for some constant $c \in \mathbb{R}$. Show that we also have that $X_n \xrightarrow{\mathcal{L}} c$.

4. Exercise

Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $X_n \sim \text{Bin}(n, \lambda/n)$ for some $\lambda \in (0, \infty)$ and integer $n > \lambda$.

(a) For a fixed integer $k \geq 0$ and n large enough, write down $\mathbb{P}(X_n = k)$.

(b) Show that $\lim_{n\to\infty} \mathbb{P}(X_n = k) = e^{-\lambda}\lambda^k/k!$ for all $k \in \{0, 1, \dots\}$.

(c) Show that if $(X_n)_{n\geq 1}$ is a sequence of random variables and X is a random variable with $X_n \in \{0, 1, \dots\}$ and $X \in \{0, 1, \dots\}$, then

$$X_n \xrightarrow{\mathcal{L}} X \iff \mathbb{P}(X_n = k) \xrightarrow{n\to\infty} \mathbb{P}(X = k) \quad \forall\, k \in \{0, 1, \dots\}.$$

(d) What do you conclude from (b)?

5. Exercise

It costs one dollar to play a certain slot machine in Las Vegas. The machine is set by the house to pay two dollars with probability 0.45 and pay nothing with probability 0.55. Let $X_i =$ the house's net winning on the i^{th} play of the machine and let $S_n = \sum_{i=1}^{n} X_i$ be the house's winning after n plays. We assume that $X_1, \ldots, X_n$ are independent.

(a) Find $\mathbb{E}[S_n]$.

(b) Find $\mathrm{Var}(S_n)$.

(c) Use the normal approximation to approximately compute $\mathbb{P}(800 < S_{10000} \leq 1100)$.