Sheet 7

Due: To be handed in before 21.04.2023 at 12:00.

1. Exercise

Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $X_n(\omega) \nearrow$ for $\forall \omega \in \Omega$ and $X_n(\omega) \geq 0$. Set $X_\infty(\omega) = \lim_{n\to\infty} X_n(\omega)$.

The Beppo Levi's Theorem says that

$$\mathbb{E}(X_\infty) = \lim_{n\to\infty} \mathbb{E}(X_n).$$

Use this result to show that if X is a random variable, then

$$\mathbb{E}(|X|) = 0 \Leftrightarrow \mathbb{P}(X = 0) = 1.$$

Hint:

- Consider $Y_n = |X|\mathbb{1}_{\{|X|\leq n\}}$ for "$\Leftarrow$".
- Write $\{X = 0\} = \bigcap_{n=1}^{\infty} \{|X| \leq \frac{1}{n}\}$ for "$\Rightarrow$".

Solution:

"$\Leftarrow$": $Y_n(\omega) \nearrow$ for $\forall \omega \in \Omega$ and $Y_n(\omega) \geq 0$. Also, $\lim_{n\to\infty} Y_n(\omega) = |X(\omega)|$.

By the Beppo Levi's Theorem, we have that $\mathbb{E}(|X|) = \lim_{n\to\infty} \mathbb{E}(Y_n)$ where

$$\begin{aligned}
\mathbb{E}(Y_n) &= \int_\Omega |X(\omega)|\mathbb{1}_{\{|X(\omega)|\leq n\}} d\mathbb{P}(\omega) \\
&= \int_{\{\omega: X(\omega)=0\}} |X(\omega)|\mathbb{1}_{\{|X(\omega)|\leq n\}} d\mathbb{P}(\omega) + \int_{\{\omega: X(\omega)\neq 0\}} |X(\omega)|\mathbb{1}_{\{|X(\omega)|\leq n\}} d\mathbb{P}(\omega) \\
&= \int_{\{\omega: X(\omega)\neq 0\}} |X(\omega)|\mathbb{1}_{\{|X(\omega)|\leq n\}} d\mathbb{P}(\omega) \\
&\leq n \int_{\{\omega: X(\omega)\neq 0\}} d\mathbb{P}(\omega) = n \cdot \mathbb{P}(X \neq 0) = n \cdot 0 = 0,
\end{aligned}$$

since $\mathbb{P}(X \neq 0) = 1 - \mathbb{P}(X = 0) = 0$. Therefore $\mathbb{E}(Y_n) = 0$, which implies that $\mathbb{E}(|X|) = 0$.

"$\Rightarrow$": $\{X = 0\} = \bigcap_{n=1}^{\infty} \{|X| \leq \frac{1}{n}\}$.

Since $\{|X| \leq \frac{1}{n}\}_{n\geq 1} \searrow$, it follows that

$$\mathbb{P}(X = 0) = \lim_{n\to\infty} \mathbb{P}\Big(|X| \leq \frac{1}{n}\Big).$$

Now,

$$\begin{aligned}
\mathbb{P}\Big(|X| \leq \frac{1}{n}\Big) &= 1 - \mathbb{P}\Big(|X| > \frac{1}{n}\Big) \\
&\geq 1 - n \cdot \mathbb{E}(|X|) \ (\textit{by Markov's inequality}) \\
&= 1.
\end{aligned}$$

$\Rightarrow \mathbb{P}\Big(|X| \leq \frac{1}{n}\Big) = 1 \Rightarrow \mathbb{P}(X = 0) = 1.$

2. Exercise

(a) Let $(X_n)_{n\geq 1}$ be a sequence of random variables and X be a random variable, all defined on the same probability space. We write that $X_n \xrightarrow{r} X$ or $X_n \xrightarrow{L_r} X$ for $r > 0$ if $\lim_{n\to\infty} \mathbb{E}[|X - X_n|^r] = 0$. Show that $X_n \xrightarrow{r} X$ implies $X_n \xrightarrow{\mathbb{P}} X$.

(b) Give an example of a sequence $(X_n)_{n\geq 1}$ such that $X_n \xrightarrow{\mathbb{P}} 0$ but not $X_n \xrightarrow{L_2} 0$.

(c) Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $\mathbb{P}(X_n = 0) = 1 - n^{-\alpha}$ and $\mathbb{P}(X_n = \sqrt{n}) = n^{-\alpha}$ for all $n \geq 1$ and some $\alpha > 0$. Show that if $\alpha > 1$, then $X_n \to 0$ a.s.

(d) Consider $Z \sim \mathcal{U}([0, 1])$ and the random sequence $(X_n)_{n\geq 1}$ defined as $X_n = \mathbb{1}_{Z\in[m2^{-k},(m+1)2^{-k})}$ if $n = 2^k + m$ with $m \in \{0, 1, \ldots , 2^k - 1\}$ and $k \in \{0, 1, \ldots \}$. Show that $X_n \xrightarrow{\mathbb{P}} 0$ but that $X_n \overset{\text{a.s.}}{\nrightarrow} 0$.

Solution:

(a) Fix $\varepsilon > 0$. Using Markov's inequality,

$$\mathbb{P}(|X - X_n| > \varepsilon) = \mathbb{P}(|X - X_n|^r > \varepsilon^r) \leq \varepsilon^{-r}\mathbb{E}[|X - X_n|^r] \xrightarrow{n\to\infty} 0. \tag{1}$$

Thus, $X_n \xrightarrow{\mathbb{P}} X$.

(b) Let

$$X_n = \begin{cases} 0 & \text{with probability } 1 - 1/n, \\ \sqrt{n} & \text{with probability } 1/n. \end{cases}$$

Fix $\varepsilon > 0$ and let $n > \varepsilon^2$. Then $\{|X_n - 0| > \varepsilon\} = \{X_n > \varepsilon\} = \{X_n = \sqrt{n}\}$, implying that $\mathbb{P}(|X_n - 0| > \varepsilon) = 1/n \to 0$ and, hence, $X_n \xrightarrow{\mathbb{P}} 0$. But $\mathbb{E}[|X_n - 0|^2] = \mathbb{E}[X_n^2] = 0 \cdot (1 - 1/n) + \sqrt{n}^2 \cdot (1/n) = 1$.

(c) Fix $\varepsilon > 0$. Then

$$\sum_{n\geq 1} \mathbb{P}(|X_n - 0| > \varepsilon) = \sum_{n\geq 1} \mathbb{P}(X_n > \varepsilon) = \sum_{n:\, n>\varepsilon^2} \mathbb{P}(X_n = \sqrt{n}) = \sum_{n:\, n>\varepsilon^2} n^{-\alpha} < \infty \quad \forall\, \alpha > 1.$$

By a result from the lecture, this implies that $X_n \to 0$ a.s.

(d) Fix $\varepsilon > 0$. Then

$$\begin{aligned}\mathbb{P}(|X_n - 0| > \varepsilon) = \mathbb{P}(X_n > \varepsilon) &= \begin{cases} 0 & \text{if } \varepsilon \geq 1, \\ \mathbb{P}(Z \in [m2^{-k}, (m + 1)2^{-k})) & \text{if } \varepsilon < 1 \end{cases} \\ &= \begin{cases} 0 & \text{if } \varepsilon \geq 1, \\ 2^{-k} = \frac{2}{2^{k+1}} \leq \frac{2}{n+1} & \text{if } \varepsilon < 1 \end{cases}\end{aligned}$$

since $n = 2^k + m \leq 2^{k+1} - 1$. Thus, $\lim_{n\to\infty} \mathbb{P}(|X_n - 0| > \varepsilon) = 0$ and $X_n \xrightarrow{\mathbb{P}} 0$.

Now, we show that $X_n \overset{\text{a.s.}}{\nrightarrow} 0$. Note that if $(x_n)_{n\geq 1}$ is some real sequence such that $\lim_{n\to 0} x_n = 0$, then we also have $\lim_{k\to 0} \max_{m\in\{0,\ldots,2^k-1\}} |x_{2^k+m}| = 0$. Indeed, for all $\varepsilon > 0$ there exists $n_0 > 0$ such that $|x_n| < \varepsilon$ for all $n \geq n_0$. Take k such that $2^k \geq n_0$. Then $|x_{2^k+m}| < \varepsilon$ for all $m \in \{0, \ldots , 2^k - 1\}$ and, hence, $\max_{m\in\{0,\ldots,2^k-1\}} |x_{2^k+m}| < \varepsilon$. Thus,

$$\begin{aligned}\mathbb{P}\Big(\lim_{n\to\infty} X_n = 0\Big) &\leq \mathbb{P}\Big(\lim_{k\to\infty} \max_{m\in\{0,\ldots,2^k-1\}} X_{2^k+m} = 0\Big) \\ &= \mathbb{P}\Big(\lim_{k\to\infty} \max_{m\in\{0,\ldots,2^k-1\}} \mathbb{1}_{Z\in[m2^{-k},(m+1)2^{-k})} = 0\Big) = \mathbb{P}(Z = 1) = 0,\end{aligned}$$

where we used that for all $z \in [0, 1)$ we have $\max_{m\in\{0,\ldots,2^k-1\}} \mathbb{1}_{z\in[m2^{-k},(m+1)2^{-k})} = 1$. This shows that $X_n \overset{\text{a.s.}}{\nrightarrow} 0$.

3. Exercise

Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $X_n \xrightarrow{\mathbb{P}} c$ for some constant $c \in \mathbb{R}$. Show that we also have that $X_n \xrightarrow{\mathcal{L}} c$.

Solution:

Let g be a continuous and bounded function on $\mathbb{R}$. We want to show that $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(c)] = g(c)$ as $n \to \infty$. Let $\varepsilon > 0$. We have that

$$\mathbb{E}[|g(X_n) - g(c)|] = \mathbb{E}[|g(X_n) - g(c)|\mathbb{1}_{|X_n - c| > \varepsilon}] + \mathbb{E}[|g(X_n) - g(c)|\mathbb{1}_{|X_n - c| \leq \varepsilon}] =: A + B.$$

Since g is continuous at c, it follows that for all $\eta > 0$ there exists $a > 0$ such that $|x - c| \leq a$ implies $|g(x) - g(c)| \leq \eta$. For $\varepsilon = a$, we have $B \leq \eta\mathbb{P}(|X_n - c| \leq \varepsilon) \leq \eta$. Also,

$$A \leq 2 \sup_{t\in\mathbb{R}} |g(t)| \; \mathbb{E}[\mathbb{1}_{|X_n - c| > a}] = 2 \sup_{t\in\mathbb{R}} |g(t)| \; \mathbb{P}(|X_n - c| > a).$$

Thus,

$$\mathbb{E}[|g(X_n) - g(c)|] \leq \eta + 2 \sup_{t\in\mathbb{R}} |g(t)| \; \mathbb{P}(|X_n - c| > a)$$

and, hence, $\limsup_{n\to\infty} \mathbb{E}[|g(X_n) - g(c)|] \leq \eta$, using the assumption that $X_n \xrightarrow{\mathbb{P}} c$. Since $\eta > 0$ was arbitrary, it follows that $\limsup_{n\to\infty} \mathbb{E}[|g(X_n) - g(c)|] = 0$, which implies $\lim_{n\to\infty} \mathbb{E}[g(X_n)] = g(c)$.

4. Exercise

Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $X_n \sim \text{Bin}(n, \lambda/n)$ for some $\lambda \in (0, \infty)$ and integer $n > \lambda$.

(a) For a fixed integer $k \geq 0$ and n large enough, write down $\mathbb{P}(X_n = k)$.

(b) Show that $\lim_{n\to\infty} \mathbb{P}(X_n = k) = e^{-\lambda}\lambda^k/k!$ for all $k \in \{0, 1, \dots\}$.

(c) Show that if $(X_n)_{n\geq 1}$ is a sequence of random variables and X is a random variable with $X_n \in \{0, 1, \dots\}$ and $X \in \{0, 1, \dots\}$, then

$$X_n \xrightarrow{\mathcal{L}} X \quad \iff \quad \mathbb{P}(X_n = k) \xrightarrow{n\to\infty} \mathbb{P}(X = k) \quad \forall\, k \in \{0, 1, \dots\}.$$

(d) What do you conclude from (b)?

Solution:

(a) For $n \geq k$, we have that $\mathbb{P}(X_n = k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$.

(b) We have

$$\begin{aligned}\mathbb{P}(X_n = k) &= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &= \underbrace{\frac{n(n-1)\cdot \dots \cdot (n-k+1)}{n^k} \left(1 - \frac{\lambda}{n}\right)^{-k}}_{\to 1 \text{ as } n\to\infty} \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \xrightarrow{n\to\infty} \frac{\lambda^k}{k!} e^{-\lambda}.\end{aligned}$$

(c) Suppose $\mathbb{P}(X_n = k) \xrightarrow{n\to\infty} \mathbb{P}(X = k)$ for all $k \in \{0, 1, \dots\}$ and let x be a point of continuity of the cdf of X. In the following, we write F_{X_n} and F_X for the cdfs of X_n and X, respectively. Then,

$$F_{X_n}(x) = \sum_{k=0}^{[x]} \mathbb{P}(X_n = k) \xrightarrow{n\to\infty} \sum_{k=0}^{[x]} \mathbb{P}(X = k) = F_X(x),$$

where $[x]$ is the integer part of x. Hence, $X_n \xrightarrow{\mathcal{L}} X$.

Now, suppose $X_n \xrightarrow{\mathcal{L}} X$. Then, for all $k \in \{0, 1, \dots\}$ and $x \in (k, k+1)$, $F_{X_n}(x) \xrightarrow{n\to\infty} F_X(x)$. Thus, $F_{X_n}(k + 1/2) \xrightarrow{n\to\infty} F_X(k + 1/2)$. But $F_{X_n}(k + 1/2) = F_{X_n}(k)$ and $F_X(k + 1/2) = F_X(k)$, which implies $F_{X_n}(k) \xrightarrow{n\to\infty} F_X(k)$. Hence,

$$\mathbb{P}(X_n = k) = F_{X_n}(k) - F_{X_n}(k-1) \xrightarrow{n\to\infty} F_X(k) - F_X(k-1) = \mathbb{P}(X = k)$$

for all $k \in \{1, 2, \dots\}$ and

$$\mathbb{P}(X_n = 0) = F_{X_n}(0) \xrightarrow{n\to\infty} F_X(0) = \mathbb{P}(X = 0).$$

(d) That $\text{Bin}(n, \lambda/n) \xrightarrow{\mathcal{L}} \text{Pois}(\lambda)$.

5. Exercise

It costs one dollar to play a certain slot machine in Las Vegas. The machine is set by the house to pay two dollars with probability 0.45 and pay nothing with probability 0.55. Let X_i = the house's net winning on the i^{th} play of the machine and let $S_n = \sum_{i=1}^n X_i$ be the house's winning after n plays. We assume that $X_1, \dots, X_n$ are independent.

(a) Find $\mathbb{E}[S_n]$.

(b) Find $\text{Var}(S_n)$.

(c) Use the normal approximation to approximately compute $\mathbb{P}(800 < S_{10000} \leq 1100)$.

Solution:

(a) $\mathbb{E}[S_n] = n\mathbb{E}[X_1] = n((-1) \cdot 0.45 + 1 \cdot 0.55) = 0.1n$. Note that we do not use independence when computing $\mathbb{E}[S_n]$.

(b) We have

$$\begin{aligned} \text{Var}(S_n) &= n\text{Var}(X_1) = n\mathbb{E}[(X_1 - 0.1)^2] \\ &= n((-1 - 0.1)^2 \cdot 0.45 + (1 - 0.1)^2 \cdot 0.55) = 0.99n. \end{aligned}$$

(c) With $n = 10000$ and $Z_n = \frac{S_n - \mathbb{E}[S_n]}{\sqrt{\text{Var}(S_n)}}$, we have

$$\begin{aligned} \mathbb{P}(800 < S_n \leq 1100) &= \mathbb{P}\left(\frac{800 - 0.1 \cdot 10000}{\sqrt{0.99 \cdot 10000}} < Z_n \leq \frac{1100 - 0.1 \cdot 10000}{\sqrt{0.99 \cdot 10000}}\right) \\ &\approx \Phi(1.005) - \Phi(-2.01) \approx 0.82 \end{aligned}$$

with Φ the cdf of $\mathcal{N}(0, 1)$.