Sheet 8

Due: To be handed in before 05.05.2023 at 12:00.

1. Exercise

Let $X_1, \dots, X_n$ be i.i.d. $\sim \mathcal{N}(0, \sigma^2)$ with $\sigma \in \Theta = (0, \infty)$. We are interested in estimating the variance σ^2. Here, $n \geq 2$.

(a) Show that the estimator

$$\tilde{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2$$

is an unbiased estimator of σ^2 such that $\tilde{\sigma}_n^2 \xrightarrow{a.s.} \sigma^2$ as $n \to \infty$.

(b) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ for $\mu \in \mathbb{R}$, $\sigma \in (0, \infty)$. Show that $\mathbb{E}[(X-\mu)^4] = 3\sigma^4$ and deduce the expression of $\mathrm{Var}((X-\mu)^2)$.

(c) Compute the mean square error of $\tilde{\sigma}_n^2$.

(d) Consider now the usual empirical estimator

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$

Admitting that

$$\mathrm{Var}_\sigma \left(\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \right) = \frac{2(n-1)}{n^2} \sigma^4$$

if $X_1, \dots, X_n$ are i.i.d. $\sim \mathcal{N}(\mu, \sigma^2)$, compute $\mathrm{MSE}_\sigma(S_n^2)$.

(e) How do you explain that $\mathrm{MSE}_\sigma(S_n^2) > \mathrm{MSE}_\sigma(\tilde{\sigma}_n^2)$?

2. Exercise

(a) Let $X_1, \dots, X_n$ be i.i.d. $\sim \mathcal{U}([0, \theta])$ with $\theta \in \Theta = (0, \infty)$. We consider the following estimators of θ.

(i) $T_1(X_1, \dots, X_n) = 2\bar{X}_n$.
(ii) $T_2(X_1, \dots, X_n) = \max_{1 \leq i \leq n} X_i$.
(iii) $T_3(X_1, \dots, X_n) = \frac{n+1}{n} \max_{1 \leq i \leq n} X_i$.

Compute the mean square error of each of the estimators and indicate whether they are unbiased or not.

<u>Hint:</u> Revisit exercises 3b) and 4c) from sheet number 5.

(b) Which estimator would you use?