Sheet 8

Due: To be handed in before 05.05.2023 at 12:00.

1. Exercise

Let $X_1, \dots, X_n$ be i.i.d. $\sim \mathcal{N}(0, \sigma^2)$ with $\sigma \in \Theta = (0, \infty)$. We are interested in estimating the variance σ^2. Here, $n \geq 2$.

(a) Show that the estimator

$$\tilde{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2$$

is an unbiased estimator of σ^2 such that $\tilde{\sigma}_n^2 \xrightarrow{a.s.} \sigma^2$ as $n \to \infty$.

(b) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ for $\mu \in \mathbb{R}$, $\sigma \in (0, \infty)$. Show that $\mathbb{E}[(X - \mu)^4] = 3\sigma^4$ and deduce the expression of $\operatorname{Var}((X - \mu)^2)$.

(c) Compute the mean square error of $\tilde{\sigma}_n^2$.

(d) Consider now the usual empirical estimator

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$

Admitting that

$$\operatorname{Var}_\sigma \left(\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \right) = \frac{2(n-1)}{n^2} \sigma^4$$

if $X_1, \dots, X_n$ are i.i.d. $\sim \mathcal{N}(\mu, \sigma^2)$, compute $\operatorname{MSE}_\sigma(S_n^2)$.

(e) How do you explain that $\operatorname{MSE}_\sigma(S_n^2) > \operatorname{MSE}_\sigma(\tilde{\sigma}_n^2)$?

Solution:

(a) Note that $\mathbb{E}_\sigma(\tilde{\sigma}_n^2) = \frac{1}{n} n \mathbb{E}_\sigma[X_1^2] = \operatorname{Var}_\sigma(X_1) + \mathbb{E}_\sigma[X_1]^2 = \sigma^2$. Since $\mathbb{E}_\sigma[X_1^2] < \infty$, it follows from the strong law of large numbers that $\tilde{\sigma}_n^2 \xrightarrow{a.s.} \mathbb{E}_\sigma[X_1^2] = \sigma^2$ (note that we used the fact the $X_1^2, \dots, X_n^2$ are i.i.d. random variables).

(b) Note that $\mathbb{E}\big[\big(\frac{X-\mu}{\sigma}\big)^4\big] = \mathbb{E}[Z^4]$ with $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. Thus,

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] &= \mathbb{E}[Z^4] = \int_{-\infty}^{\infty} z^4 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= -z^3 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \Big|_{-\infty}^{\infty} + 3 \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= 3\mathbb{E}[Z^2] = 3\operatorname{Var}(Z) = 3.
\end{aligned}$$

It follows that $\mathbb{E}[(X-\mu)^4] = 3\sigma^4$. Thus,

$$\operatorname{Var}((X-\mu)^2) = \mathbb{E}[(X-\mu)^4] - \mathbb{E}[(X-\mu)^2]^2 = 3\sigma^4 - \sigma^4 = 2\sigma^4.$$

(c) We calculate

$$\operatorname{MSE}_\sigma(\tilde{\sigma}_n^2) = \operatorname{bias}_\sigma^2(\tilde{\sigma}_n^2) + \operatorname{Var}_\sigma(\tilde{\sigma}_n^2) = \operatorname{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\operatorname{Var}(X_1^2) = \frac{2\sigma^4}{n}.$$

(d) Put $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2$. Then, $S_n^2 = \frac{n}{n-1}\hat{\sigma}_n^2$ and, hence,

$$\mathrm{MSE}_\sigma(S_n^2) = \mathrm{bias}_\sigma^2(S_n^2) + \mathrm{Var}_\sigma(S_n^2) = \mathrm{bias}_\sigma^2(S_n^2) + \frac{n^2}{(n-1)^2}\mathrm{Var}_\sigma(\hat{\sigma}_n^2).$$

Thus,

$$\mathrm{MSE}_\sigma(S_n^2) = \mathrm{bias}_\sigma^2(S_n^2) + \frac{n^2}{(n-1)^2}\frac{2(n-1)}{n^2}\sigma^4 = \mathrm{bias}_\sigma^2(S_n^2) + \frac{2}{(n-1)}\sigma^4.$$

Furthermore,

$$\begin{aligned}
\mathbb{E}_\sigma[S_n^2] &= \frac{1}{n-1}\mathbb{E}_\sigma\left[\sum_{i=1}^n (X_i - \bar{X}_n)^2\right] \\
&= \frac{1}{n-1}\mathbb{E}_\sigma\left[\sum_{i=1}^n (X_i-\mu)^2 + 2\sum_{i=1}^n (X_i-\mu)(\mu-\bar{X}_n) + n(\mu-\bar{X}_n)^2\right] \\
&= \frac{1}{n-1}\mathbb{E}_\sigma\left[\sum_{i=1}^n (X_i-\mu)^2 - n(\mu-\bar{X}_n)^2\right] \\
&= \frac{1}{n-1}\left(\sum_{i=1}^n \mathbb{E}_\sigma[(X_i-\mu)^2] - n\mathrm{Var}_\sigma(\bar{X}_n)\right) \\
&= \frac{1}{n-1}\big(n\mathbb{E}_\sigma[(X_1-\mu)^2] - \mathrm{Var}_\sigma(X_1)\big) = \mathrm{Var}_\sigma(X_1) = \sigma^2.
\end{aligned}$$

This means that $\mathrm{bias}_\sigma^2(S_n^2) = 0$. Thus, $\mathrm{MSE}_\sigma(S_n^2) = \frac{2}{n-1}\sigma^4$.

(e) For a fixed $n \geq 2$, $\mathrm{MSE}_\sigma(\tilde{\sigma}_n^2) < \mathrm{MSE}_\sigma(S_n^2)$ since $\frac{1}{n} < \frac{1}{n-1}$. This is to be expected because the estimator $\tilde{\sigma}_n^2$ uses the additional information that the variables $X_1, \dots, X_n$ have expectation zero, while in S_n^2 we "ignore" it and estimate it using $\bar{X}_n$.

2. Exercise

(a) Let $X_1, \dots, X_n$ be i.i.d. $\sim \mathcal{U}([0,\theta])$ with $\theta \in \Theta = (0, \infty)$. We consider the following estimators of θ.

(i) $T_1(X_1, \dots, X_n) = 2\bar{X}_n$.
(ii) $T_2(X_1, \dots, X_n) = \max_{1 \leq i \leq n} X_i$.
(iii) $T_3(X_1, \dots, X_n) = \frac{n+1}{n}\max_{1 \leq i \leq n} X_i$.

Compute the mean square error of each of the estimators and indicate whether they are unbiased or not.

Hint: Revisit exercises 3b) and 4c) from sheet number 5.

(b) Which estimator would you use?

Solution:

(a) (i) We have $\mathbb{E}_\theta(2\bar{X}_n) = 2\mathbb{E}_\theta(X_1) = \theta$. This means that $T_1(X_1, \dots, X_n)$ is unbiased. Further,

$$\mathrm{MSE}_\theta(T_1) = \mathrm{Var}_\theta(T_1) = \mathrm{Var}_\theta(2\bar{X}_n) = \frac{4}{n}\mathrm{Var}_\theta(X_1) = \frac{\theta^2}{3n}.$$

(ii) We know from sheet number 5 that $\max_{1\leq i\leq n} \frac{X_i}{\theta} \sim \text{Beta}(n,1)$ since $\frac{X_i}{\theta} \sim \mathcal{U}([0,1])$. Therefore,

$$\mathbb{E}_\theta(T_2) = \mathbb{E}_\theta\left[\max_{1\leq i\leq n} \frac{X_i}{\theta}\right]\theta = \frac{n}{n+1}\theta$$
$$\text{Var}_\theta(T_2) = \text{Var}_\theta\left(\max_{1\leq i\leq n} \frac{X_i}{\theta}\right)\theta^2 = \frac{n}{(n+1)^2(n+2)}\theta^2.$$

Thus, $\text{bias}_\theta(T_2) = \frac{n}{n+1}\theta - \theta = -\frac{\theta}{n+1} \neq 0$, so T_2 is biased, and

$$\text{MSE}_\theta(T_2) = \frac{\theta^2}{(n+1)^2} + \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{\theta^2}{(n+1)^2}\left(1 + \frac{n}{n+2}\right) = \frac{2\theta^2}{(n+1)(n+2)}.$$

(iii) We have $\mathbb{E}_\theta(T_3) = \frac{n+1}{n}\mathbb{E}_\theta(T_2) = \theta$, so T_3 is unbiased. Further,

$$\text{MSE}_\theta(T_3) = \text{Var}_\theta(T_3) = \frac{(n+1)^2}{n^2}\text{Var}_\theta(T_2) = \frac{\theta^2}{n(n+2)}.$$

(b) We have

$$\text{MSE}_\theta(T_1) > \text{MSE}_\theta(T_2) > \text{MSE}_\theta(T_3)$$

for all $\theta \in \Theta$, $n > 1$. The first inequality is clear; the second one is

$$\text{MSE}_\theta(T_2) - \text{MSE}_\theta(T_3) = \left(\frac{2}{(n+1)(n+2)} - \frac{1}{n(n+2)}\right)\theta^2$$
$$= \left(\frac{2}{n+1} - \frac{1}{n}\right)\frac{\theta^2}{n+2} = \frac{(n-1)\theta^2}{n(n+1)(n+2)} > 0.$$

In conclusion, we would use the estimator T_3.