Sheet 9

Due: To be handed in before 05.05.2023 at 12:00.

1. Exercise

Let $X_1, \ldots, X_n$ be i.i.d. random variables $\sim$ Bernoulli(θ_0) for some unknown $\theta_0 \in \Theta = (0, 1)$.

(a) Show that the moment estimator of θ_0 is $\bar{X}_n$.

(b) Using the Central Limit Theorem, state the asymptotic distribution of $\hat{\theta}$.

(c) Using the weak law of large numbers and Slutsky's theorem, find a bilateral and symmetric confidence interval for θ_0 of asymptotic level equal to 95%.

Solution:

(a) For $\theta \in \Theta$, $\mu_1(\theta) = \mathbb{E}_\theta[X] = \theta$ if $X \sim P_\theta$ and P_θ is the distribution of Bernoulli(θ). Then, replacing the theoretical first moment by its empirical (or sample) version, we get $\hat{\theta} = \bar{X}_n$.

(b) Since $X \sim$ Bernoulli(θ_0) has $\text{Var}_{\theta_0}(X) < \infty$, it follows from the CLT that

$$\frac{\sqrt{n}(\hat{\theta} - \theta_0)}{\sqrt{\theta_0(1-\theta_0)}} = \frac{\sqrt{n}(\hat{\theta} - \theta_0)}{\sqrt{\text{Var}_{\theta_0}(X)}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1).$$

(c) By the WLLN, we have $\bar{X}_n \xrightarrow{\mathbb{P}} \theta_0$. By the continuous mapping theorem (lecture), we have

$$\sqrt{\frac{\theta_0(1-\theta_0)}{\bar{X}_n(1-\bar{X}_n)}} \xrightarrow{\mathbb{P}} 1.$$

This follows from continuity of the function $f(x) = \sqrt{\frac{\theta_0(1-\theta_0)}{x(1-x)}}$, $x \in (0, 1)$. Thus,

$$\frac{\sqrt{n}(\hat{\theta} - \theta_0)}{\sqrt{\bar{X}_n(1-\bar{X}_n)}} = \frac{\sqrt{n}(\bar{X}_n - \theta_0)}{\sqrt{\theta_0(1-\theta_0)}} f(\bar{X}_n) \xrightarrow{d} Z$$

by Slutsky's theorem. For any $\alpha \in (0, 1)$, let $z_{1-\alpha/2}$ be the $(1-\alpha/2)$-quantile of $\mathcal{N}(0, 1)$. Then,

$$\mathbb{P}\left(-z_{1-\alpha/2} \le \frac{\sqrt{n}(\bar{X}_n - \theta_0)}{\sqrt{\bar{X}_n(1-\bar{X}_n)}} \le z_{1-\alpha/2}\right) \approx \Phi(z_{1-\alpha/2}) - \Phi(-z_{1-\alpha/2}) = 2\Phi(z_{1-\alpha/2}) - 1 = 1 - \alpha,$$

equivalently

$$\mathbb{P}\left(\theta_0 \in \left[\bar{X}_n - \frac{z_{1-\alpha/2}}{\sqrt{n}}\sqrt{\bar{X}_n(1-\bar{X}_n)}, \bar{X}_n + \frac{z_{1-\alpha/2}}{\sqrt{n}}\sqrt{\bar{X}_n(1-\bar{X}_n)}\right]\right) \approx 1 - \alpha.$$

If $\alpha = 0.95$, then $z_{1-\alpha/2} = z_{0.975} \approx 1.96$ and

$$\left[\bar{X}_n - \frac{1.96}{\sqrt{n}}\sqrt{\bar{X}_n(1-\bar{X}_n)}, \bar{X}_n + \frac{1.96}{\sqrt{n}}\sqrt{\bar{X}_n(1-\bar{X}_n)}\right]$$

is a bilateral symmetric confidence interval for θ_0 of asymptotic level $\approx 95\%$.

2. Exercise

Let $X_1, \ldots, X_n$ be i.i.d. $\sim \mathcal{U}(0, \theta_0)$ for some $\theta_0 \in \Theta = (0, \infty)$. For $X \sim \mathcal{U}(0, \theta_0)$, we adopt the following expression for the density: $p_\theta(x) = \frac{1}{\theta} \mathbb{1}_{0 \le x \le \theta}$.

(a) Write the likelihood based on the sample $\mathbb{X} = (X_1, \ldots, X_n)$ and show that the MLE is $\hat{\theta} = \max_{1 \le i \le n} X_i$.

(b) Show that $\hat{\theta} \xrightarrow{\mathbb{P}} \theta_0$.

<u>Hint:</u> We can first show that $\mathbb{E}_{\theta_0}[|\hat{\theta} - \theta_0|] \to 0$ as $n \to \infty$.

Solution:

(a) We have

$$L_{\mathbb{X}}(\theta) = \frac{1}{\theta^n} \prod_{i=1}^n \mathbb{1}_{0 \le X_i \le \theta} = \frac{1}{\theta^n} \prod_{i=1}^n \mathbb{1}_{0 \le X_i} \mathbb{1}_{X_i \le \theta} = \frac{1}{\theta^n} \mathbb{1}_{0 \le \min_i X_i} \mathbb{1}_{\max_i X_i \le \theta}.$$

Then, maximizing $\theta \mapsto L_{\mathbb{X}}(\theta)$ is equivalent to maximizing the function $\theta \mapsto \frac{1}{\theta^n} \mathbb{1}_{\max_i X_i \le \theta}$. This shows clearly that $\hat{\theta} = \max_{1 \le i \le n} X_i$ is the MLE.

(b) First, note that $\hat{\theta} \le \theta_0$ almost surely. Indeed,

$$\mathbb{P}_{\theta_0}(\hat{\theta} > \theta_0) = \mathbb{P}_{\theta_0}(\exists\, i \in \{1, \ldots, n\} \colon X_i > \theta_0) \le \sum_{i=1}^n \mathbb{P}_{\theta_0}(X_i > \theta_0) = n\mathbb{P}_{\theta_0}(X_1 > \theta_0) = 0.$$

Then,

$$\mathbb{E}_{\theta_0}[|\hat{\theta} - \theta_0|] = \mathbb{E}_{\theta_0}[\theta_0 - \hat{\theta}] = \theta_0 \left(1 - \mathbb{E}_{\theta_0}\left[\frac{\hat{\theta}}{\theta_0}\right]\right).$$

We have seen that $\frac{X_1}{\theta_0}, \ldots, \frac{X_n}{\theta_0}$ are i.i.d. $\sim \mathcal{U}(0,1)$ and that $\max_{1 \le i \le n} \frac{X_i}{\theta_0} \sim \text{Beta}(n, 1)$. Thus,

$$\mathbb{E}_{\theta_0}\left[\max_{1 \le i \le n} \frac{X_i}{\theta_0}\right] = \frac{n}{n+1}.$$

It follows that $1 - \mathbb{E}_{\theta_0}[\hat{\theta}/\theta_0] = 1 - \frac{n}{n+1} = \frac{1}{n+1} \to 0$. Now, by Markov's inequality,

$$\mathbb{P}_{\theta_0}(|\hat{\theta} - \theta_0| > \varepsilon) \le \frac{1}{\varepsilon} \mathbb{E}_{\theta_0}[|\hat{\theta} - \theta_0|] \to 0$$

for all $\varepsilon > 0$, hence $\hat{\theta} \xrightarrow{\mathbb{P}} \theta_0$.