Exercise Sheet 1

Algebraic Topology II

09.03.2023

Q1 Consider the space $X := \Delta^n/\partial\Delta^n$ for $n \geq 0$. Denote by $* \in X$ the point corresponding to $\partial\Delta^n$. The quotient map $\sigma_n : \Delta \to X$, viewed as a singular n-simplex, is a cycle in $S_n(X,*)$.

(a) Show that $[\sigma_n]$ generates $H_n(X,*;\mathbb{Z}) \cong \widetilde{H}_n(X;\mathbb{Z}) \cong \mathbb{Z}$.

(b) Let G be an abelian group. Then for all $g \in G$, $g \cdot \sigma_n$ is a cycle in $S_n(X,*;G)$. Show that the map

$$G \to H_n(X,*;G), g \mapsto [g \cdot \sigma_n]$$

is an isomorphism.

Q2 Let $\pi : X \to Y$ be a $2:1$ covering. Recall the short exact sequence of chain complexes

$$0 \to S_\bullet(Y,\mathbb{Z}_2) \xrightarrow{T} S_\bullet(X;\mathbb{Z}_2) \xrightarrow{\pi_c} S_\bullet(Y;\mathbb{Z}_2) \to 0$$

and its associated long exact sequence in homology. These sequences are called the *Gysin* sequence.

Show that

(a) $T \circ \pi_c = \mathrm{id} + \Theta_c$, where $\Theta : X \to X$ is the unique non-trivial desk transformation of π.

(b) Assume that $H_i(X,\mathbb{Z}_2) = \mathbb{Z}_2$ for some i. Shwo that $T_* \circ \pi_* = 0$ in degree i.

Q3 Suppose that you know $H_k(\mathbb{R}P^n,\mathbb{Z}_2) = 0$ for all $k > n$. Use the Gysin sequence for this covering to compute $H_k(\mathbb{R}P^n;\mathbb{Z}_2)$ for $0 \leq k \leq n$.

Q4 Show that $\mathbb{R}P^2$ is not a retract of $\mathbb{R}P^3$.

Q5 Use Borsuk-Ulam to prove that whenever there exists a map $\phi : S^n \to S^m$ which is equivariant with respect to the antipodal maps, then $n \leq m$.

Q6 Does the Borsuk-Ulam hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exists $(x,y) \in S^1 \times S^1$ such that $f(x,y) = f(-x,-y)$?

Q7 Consider $\mathbb{R}P^k = S^k/(x \sim -x)$ and denote by $q : S^k \to \mathbb{R}P^k$ the quotient map. View $\mathbb{R}P^{k-1}$ as a subspace of $\mathbb{R}P^k$ as follows: let $S^{k-1}_{E_q} \subset S^k$ be the equator

$$S^{k-1}_{E_q} = \{(x_1,\cdots,x_{k+1}) \in S^k : x_{k+1} = 0\} \subset S^k\,.$$

Then $q(S^{k-1}_{E_q}) \subset \mathbb{R}P^k$ is homeomorphic to $\mathbb{R}P^{k-1}$. Consider the space $\mathbb{R}P^k/\mathbb{R}P^{k-1}$ and the quotient map $q' : \mathbb{R}P^k \to \mathbb{R}P^k/\mathbb{R}P^{k-1}$. Denote by

$$B^k_+ := \{(x_1,\cdots,x_{k+1}) \in S^k : x_{k+1} \geq 0\} \subset S^k$$

the closed upper hemisphere and similarly by B^k_- the closed lower hemisphere.

(a) Show that there exists a homeomorphism $\phi : \mathbb{R}P^k/\mathbb{R}P^{k-1} \to S^k$ such that the composition of maps

$$f := \left(S^k \xrightarrow{q} \mathbb{R}P^k \xrightarrow{q'} \mathbb{R}P^k/\mathbb{R}P^{k-1} \xrightarrow{\phi} S^k \right)$$

sends each open hemisphere $\mathrm{Int}(B^k_\pm) \subset S^k$ homeomorphically onto $S^k \setminus \{\text{point}\}$.

(b) Show that $\deg(f) = \pm(1 + (-1)^{k-1})$. (The $\pm$ depends on the choice of ϕ.)
Hint: Use local degrees.

(c) Consider the space

$$\mathbb{R}P^k \cup_{h_\partial} B^{k+1},$$

where the attaching map $h_\partial : \partial B^{k+1} = S^k \to \mathbb{R}P^k$ is the quotient map q. Show that there exists a homeomorphism

$$(\mathbb{R}P^k \cup_{h_\partial} B^{k+1}, \mathbb{R}P^k) \approx (\mathbb{R}P^{k+1}, \mathbb{R}P^k)$$

which is the identity on $\mathbb{R}P^k$.

(d) Endow $\mathbb{R}P^n$ with the structure of an n-dimensional CW-complex X with one j-cell in each dimension $0 \le j \le n$, as follows:

$$\begin{aligned}
&X^{(0)} = \mathbb{R}P^0 = \text{ 1 point},\\
&\dots\\
&X^{(k)} \approx \mathbb{R}P^k,\\
&X^{(k+1)} \approx \mathbb{R}P^k \cup_{h_\partial} B^{k+1} \approx \mathbb{R}P^{k+1},\\
&\dots\\
&X^{(n)} \approx \mathbb{R}P^{n-1} \cup_{h_\partial} B^n \approx \mathbb{R}P^n.
\end{aligned}$$

(e) Consider the cellular chain complex $C^{CW}_\bullet(X)$ of the CW-complex described in (d). Denote by $e^{(k)}$ the generator of $C^{CW}_k(X)$, corresponding to the k-dimensional cell, so that $C^{CW}_k(X) = \mathbb{Z}e^{(k)}$. Calculate the differential $d : C^{CW}_{k+1}(X) \to C^{CW}_k(X)$.