Exercise Sheet 3

Algebraic Topology II

02.05.2023

Q1

1. For $G \in \{\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$, compute the ring structure for the cohomology with G-coefficients of the Klein bottle K.
2. Repeat the previous part for the connected sum of K with an oriented surface M of genus g, i.e. the space obtained by cutting out small disks from K and M, and gluing together the resulting boundaries along a degree one map.

Q2

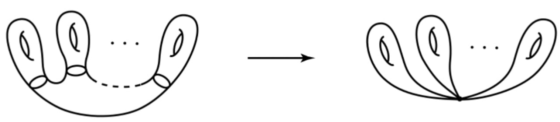
1. Compute the cup product structure on $H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$.
2. Using the cup product structure, show there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ if $n > m$. What is the corresponding result for maps $\mathbb{C}P^n \to \mathbb{C}P^m$?
3. Prove the Borsuk–Ulam theorem using part 2).

Q3 Suppose that a space X can be covered by two acyclic open sets A and B. Using the cup product

$$H^k(X, A; R) \times H^l(X, B; R) \to H^{k+l}(X, A \cup B; R),$$

show that all cup products of classes in $H^*(X; R)$ of positive dimensions vanish. Generalize to the situation that X can be covered by n acyclic open sets.

Q4 Compute the cup product structure on $H^*(\Sigma_g; \mathbb{Z})$ for the closed orientable surface Σ_g of genus g, assuming as known the cup product structure on $H^*(T^2; \mathbb{Z})$ and using the map $\pi : \Sigma_g \to \bigvee_g T^2$ depicted below.



Q5 Show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.