AUGMENTATION

Let X path connected, $x_0 \in X$. Then the augmentation is the map

$$\varepsilon_{x_0} : S_\cdot(X) \to S_\cdot(X),$$ defined by

$\varepsilon_{x_0} \equiv 0$ on $S_i(X)$ $\forall i > 0$ and

$$\varepsilon_{x_0}\Big(\sum_{x \in X} n_x x\Big) = \Big(\sum_{x \in X} n_x\Big) x_0.$$

ε_{x_0} is a chain map.

LEMMA (CONE CONSTRUCTION)

Assume X is contractible. Fix $x_0 \in X$. Then $\exists$ a chain homotopy

$$D = D_{X, x_0} : S_\cdot(X) \to S_\cdot(X)[1]$$

s.t. $D\partial + \partial D = id - \varepsilon_{x_0}$ (in particular, $H_i(X) = 0$ $\forall i > 0$.)

NOTATION

For a chain complex $C_\cdot$ denote by $C_\cdot[d]$ the same complex, but with a shift in degree

$(C[d])_i := C_{i+d}$ and the same boundary map.

If D° is cohomologically graded $(D[d])^i = D^{i-d}$.

PROOF

D is constructed using a cone-construction and a homotopy

$F : X \times [0,1] \to X$ with $F(x,0) = x$,

$F(x,1) = x_0 \quad \forall x \in X.$

Recall that one canonical construction of the n-simplex Δ_n is as the set of points

$$\Delta_n = \left\{ \sum_{i=0}^{n} t_i e_i \;\middle|\; \sum_i t_i = 1,\ t_i \geq 0\ \forall i \right\} \subseteq \mathbb{R}^{n+1}$$

where $\{e_0, \dots, e_n\}$ is the standard basis in $\mathbb{R}^{n+1}$. With this description we can regard $(t_0, t_1, \dots, t_n)$ as 'coordinates' Δ_n. In particular, the faces of Δ_n are given by $\{t_i = 0\} : i = 0, \dots, n$ and the vertices are given by $\{t_i = 1\}$.

Given a singular n-simplex $\sigma : \Delta_n \to X$, we define

$$D(\sigma)(t_0, \dots, t_{n+1}) = F\left(\sigma\left(\left(\frac{t_1, \dots, t_{n+1}}{1-t_0}\right)\right), t_0\right) : \Delta_{n+1} \longrightarrow X$$

Observe that the face $\Delta_n \cong \{t_0 = 0\} \subset \Delta_{n+1}$ is mapped onto $\sigma(\Delta_n)$ and the vertex $\{t_0 = 1\}$ is mapped onto the contraction point x_0. If σ has degree ≥ 1 one can check that

$$\partial D(\sigma) = \sigma - D(\partial\sigma).$$

and if σ has degree 0, then

$$\partial D(\sigma) = \sigma - x_0,$$

where we identify x_0 and the 0-simplex with image $x_0 \in X$ thus

$$\partial D + D\partial = \mathrm{Id} - \varepsilon$$

where ε is 0 in nonzero degrees and $\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i x_0$ can be identified with the augmentation map in degree 0. $\blacksquare$

THEOREM

$\exists$ chain map $\theta : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$, defined $\forall$ spaces X, Y, which is natural in X & Y and s.t. in degree 0 we have: $\forall x \in X, y \in Y$, $\theta((x,y)) = x \otimes y$.

Naturality means: $\forall$ maps $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ we have a commutative diagram

$$\begin{tikzcd}
S_\bullet(X \times Y) \arrow[r, "\theta"] \arrow[d, "(f\times g)_c"'] & S_\bullet(X) \otimes S_\bullet(Y) \arrow[d, "f_c \otimes g_c"] \\
S_\bullet(X' \times Y') \arrow[r, "\theta"] & S_\bullet(X') \otimes S_\bullet(Y')
\end{tikzcd}$$

Additionally, $\partial_\otimes \circ \theta = \theta \partial$.

LEMMA

Let X, Y be contractible spaces, $x_0 \in X, y_0 \in Y$. then $\exists$ a chain homotopy

$$E : S_\bullet(X) \otimes S_\bullet(Y) \to \left(S_\bullet(X) \otimes S_\bullet(Y)\right)[1]$$

between $\varepsilon_{x_0} \otimes \varepsilon_{y_0}$ and $id \otimes id$. In particular,

$$H_n\left(S_\bullet(X) \otimes S_\bullet(Y)\right) = 0 \qquad \forall n \geq 1$$

and $\forall$ 0-chain $\sum n_{x,y}\, x \otimes y$ we have

$$\sum n_{xy} [x \otimes y] = \left(\sum n_{xy}\right) [x_0 \otimes y_0].$$

EXERCISE

Let $A_\bullet \xrightarrow{f'} A_\bullet'$, $A_\bullet' \xrightarrow{f''} A_\bullet''$, $B_\bullet \xrightarrow{g'} B_\bullet'$, $B_\bullet' \xrightarrow{g''} B_\bullet''$ be graded homo. Consider homomorphisms

$$(f'' \circ f') \otimes (g'' \circ g') \ \& \ (f'' \otimes g'') \circ (f' \otimes g') : A \otimes B \to A'' \otimes B''$$

Show that

$$(f'' \circ f') \otimes (g'' \circ g') = (-1)^{|f'| \cdot |g''|} (f'' \otimes g'') \circ (f' \otimes g')$$

PROOF

We'll use the chain homotopies D_X & D_Y between $id_{S_\bullet(X)}$ & ε_{x_0} and $id_{S_\bullet(Y)}$ & ε_{y_0} coming from the fact that X & Y are contractible.

$E := D_X \otimes id + \varepsilon_{x_0} \otimes D_Y$. Recall the diff. d on $S_\bullet(X) \otimes S_\bullet(Y)$

$$d = \partial_X \otimes id + id \otimes \partial_Y$$ (we use the Koszul sign convention)

$$Ed + dE = (D_X \otimes id + \varepsilon_{x_0} \otimes D_Y) \circ (\partial_X \otimes id + id \otimes \partial_Y)$$
$$+ (\partial_X \otimes id + id \otimes \partial_Y) \circ (D_X \otimes id + \varepsilon_{x_0} \otimes D_Y)$$
$$= (D_X \circ \partial_X) \otimes id + D_X \otimes \partial_Y - (\varepsilon_{x_0} \circ \partial_X) \otimes D_Y$$
$$+ \varepsilon_{x_0} \otimes (D_Y \circ \partial_Y) + (\partial_X \circ D_X) \otimes id + (\partial_X \circ \varepsilon_{x_0}) \otimes D_Y$$
$$- D_X \otimes \partial_Y + \varepsilon_{x_0} \otimes (\partial_Y \circ D_Y) = (id - \varepsilon_{x_0}) \otimes id$$
$$+ \varepsilon_{x_0} \otimes (id - \varepsilon_{y_0}) = id \otimes id - \varepsilon_{x_0} \otimes id$$
$$+ \varepsilon_{x_0} \otimes id - \varepsilon_{x_0} \otimes \varepsilon_{y_0} =$$

$= id \otimes id \sim \varepsilon_{x_0} \otimes \varepsilon_{y_0}$

It follows that $id \otimes id$ and $\varepsilon_{x_0} \otimes \varepsilon_{y_0}$ induce the same maps on homology groups. Since ε_{x_0} is 0 in all degrees but 0, $H_i(S.(X) \otimes S.(Y)) = 0$ $\forall i > 0$. In degree 0 we get $(\varepsilon_{x_0} \otimes \varepsilon_{y_0})_* = (id)_*$, so

$$\sum n_{xy} [x \otimes y] = \left(\sum n_{xy}\right) [x_0 \otimes y_0].$$

$\blacksquare$