Prof. Paul Biran
ETH Zürich

Algebraic Topology II

FS 2021

Solutions to problem set 1

1. (a) Consider the diagram

$$\begin{array}{ccc} H_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\cong} & \widetilde{H}_{n-1}(\partial\Delta^n) \\ \downarrow \cong & & \\ H_n(\Delta^n/\partial\Delta^n, *) & & \end{array} \tag{1}$$

The horizontal map is the boundary map from the (reduced) LES for the pair $(\Delta^n, \partial\Delta^n)$, which is an isomorphism by looking at the neighbouring terms in the LES. The vertical map is induced by the quotient map $(\Delta^n, \partial\Delta^n) \to (\Delta^n/\partial\Delta^n, *)$ and is an isomorphism since $(\Delta^n, \partial\Delta^n)$ is a good pair.

Consider now the tautological n-simplex $\alpha_n : \Delta^n \to \Delta^n$, which defines a class $[\alpha_n] \in H_n(\Delta^n, \partial\Delta^n)$. The image of $[\alpha_n]$ under the vertical map is $[\sigma_n] \in H_n(\Delta^n/\partial\Delta^n, *)$, while its image under the horizontal map is the class $[\beta_{n-1}] \in \widetilde{H}_{n-1}(\partial\Delta^n)$ with

$$\beta_{n-1} = \partial_n \, \alpha_n = \sum_{i=0}^{n} (-1)^i F_i^n \in C_{n-1}(\partial\Delta^n),$$

where $F_i^n : \Delta^{n-1} \to \partial\Delta^n$ is the i-th face map of the simplex Δ^n. So once we know that $[\beta_{n-1}]$ generates $\widetilde{H}_{n-1}(\partial\Delta^n)$, we can conclude from (1) that $[\sigma_n]$ generates $H_n(\Delta^n/\partial\Delta^n, *)$.

It is clear that $[\beta_0]$ generates $\widetilde{H}_0(\partial\Delta^1)$, so we know that $[\sigma_1]$ generates $H_1(\Delta^1/\partial\Delta^1, *)$, which is what the problem asks us to prove for $n = 1$. We now proceed by induction; for the inductive step, consider the map $\phi : \partial\Delta^n \to \Delta^{n-1}/\partial\Delta^{n-1}$ which collapses all except the zero-th face to a point, and the induced map $\phi_* : H_{n-1}(\partial\Delta^n) \to H_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}, *)$. Observe that $\phi_*[\beta_{n-1}] = [\sigma_{n-1}]$; since $[\sigma_{n-1}]$ generates by inductive assumption, we conclude that $[\beta_{n-1}]$ generates.

(b) Analogous to (a). In summary, there are isomorphisms

$$\begin{array}{ccccccc} H_n(\Delta^n/\partial\Delta^n, *; G) & \underset{q_*}{\overset{\cong}{\longleftarrow}} & H_n(\Delta^n, \partial\Delta^n; G) & \underset{\cong}{\overset{\partial_*}{\longrightarrow}} & \tilde{H}_{n-1}(\partial\Delta^n; G) & \underset{\cong}{\overset{\phi_*}{\longrightarrow}} & H_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}, *; G) \\ [g\sigma_n] & \longleftarrow & [g\alpha_n] & \longrightarrow & [g\beta_{n-1}] & \longrightarrow & [g\sigma_{n-1}] \end{array}$$

and

$$\begin{aligned} G &\longrightarrow \tilde{H}_0(\partial\Delta^0; G) \\ g &\longmapsto [g\beta_0]. \end{aligned}$$

2. Consider the cover of Y given by the subsets $A = \Delta^n_+$ and $B = \Delta^n_-$. Both are contractible and we have $A \cap B = \partial\Delta^n$, so that the relevant piece of the corresponding reduced MV sequence reads

$$0 \to \widetilde{H}_n(Y) \xrightarrow{\partial_*} \widetilde{H}_{n-1}(\partial\Delta^n) \to 0$$

Note that $\partial_*[\tau_+ - \tau_-] = [\partial\tau_+] = [\beta_{n-1}] \in \widetilde{H}_{n-1}(\partial\Delta^n)$ with $\beta_{n-1} \in C_{n-1}(\partial\Delta^n)$ defined as in the solution to the previous problem. Since $[\beta_{n-1}]$ generates (see the previous problem) we deduce that $[\tau_+ - \tau_-]$ generates.

We give an alternative inductive proof that $[\beta_n]$ generates $\widetilde{H}_{n-1}(\partial\Delta^n)$ using the Mayer-Vietoris sequence. For $n = 0$ the statement is clear. For the inductive step, consider the cover of $\partial\Delta^{n+1}$ given by $A := \operatorname{im} F_0^{n+1}$ and $B := \partial\Delta^{n+1} \setminus \operatorname{int} A$ (the interiors don't cover all of $\partial\Delta^{n+1}$, but that can be repaired by taking small thickenings of A and B). Since both A and B are contractible, the corresponding reduced MV sequence splits into pieces of the form

$$0 \to \widetilde{H}_n(\partial\Delta^{n+1}) \xrightarrow{\cong} \widetilde{H}_{n-1}(A \cap B) \to 0$$

Note that we can identify $A \cap B = \partial A$ with $\partial\Delta^n$ via $F_0^{n+1}|_{\partial\Delta^n}$. By definition of the MV boundary map $\partial_* : \widetilde{H}_n(\partial\Delta^{n+1}) \to \widetilde{H}_{n-1}(A \cap B)$, we have $\partial_*[\beta_n] = [\partial F_0^{n+1}]$, which in our identification $A \cap B \cong \partial\Delta^n$ is $[\beta_{n-1}]$. Since ∂_* is an isomorphism and $[\beta_{n-1}]$ generates $\widetilde{H}_{n-1}(\partial\Delta^n)$ by inductive assumption, it follows that $[\beta_n]$ generates $\widetilde{H}_n(\partial\Delta^{n+1})$.

3. (a) Let $\tilde{\sigma}\colon \Delta^k \to X$ be a singular simplex. Then

$$T \circ \pi_c(\tilde{\sigma}) = \tilde{\sigma} + \Theta \circ \tilde{\sigma},$$

because $\tilde{\sigma}$ and $\Theta \circ \tilde{\sigma}$ are the two liftings of $\pi \circ \tilde{\sigma}$. Passing to homology, it follows that $T_* \circ \pi_* = id + \Theta_*$.

(b) If $H_i(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$, then $\Theta_*\colon H_i(X;\mathbb{Z}_2) \to H_i(X;\mathbb{Z}_2)$ is the identity. (Θ_* is an isomorphism and id is the only isomorphism on $\mathbb{Z}_2$.) So $T_* \circ \pi_* = id + id = 0$ in degree i.

4. *In the following, all homology groups have $\mathbb{Z}_2$ coefficients.* Given that $H_k(\mathbb{R}P^n) = 0$ for $k > n$ by assumption, the leftmost piece of the Smith sequence for the cover $p : S^n \to \mathbb{R}P^n$ looks like

$$0 \to H_n(\mathbb{R}P^n) \xrightarrow{t_*} H_n(S^n) \xrightarrow{p_*} H_n(\mathbb{R}P^n) \xrightarrow{\partial_*} H_{n-1}(\mathbb{R}P^n) \to H_{n-1}(S^n) = 0 \to \dots$$

Here t_* is induced by the map $C_*(\mathbb{R}P^n) \to C_*(S^n)$ taking a simplex $\sigma : \Delta^k \to \mathbb{R}P^k$ to $\tilde{\sigma} + \alpha \circ \tilde{\sigma}$, where $\tilde{\sigma} : \Delta^n \to S^n$ is one of the two possible lifts of σ to S^n and where $\alpha : S^n \to S^n$ denotes the antipodal map. Note that we have $t_* \circ p_* = (\mathrm{id} + \alpha_*) : H_*(S^n) \to H_*(S^n)$, which implies $t_* \circ p_* = 0$ because $\alpha_* = \mathrm{id} : H_*(S^n) \to H_*(S^n)$ (because α_* is an involution and $H_k(S^n)$ either vanishes or is $\mathbb{Z}_2$). This together with the fact that $t_* : H_n(\mathbb{R}P^n) \to H_n(S^n)$ is injective implies that $p_* : H_n(S^n) \to H_n(\mathbb{R}P^n)$ vanishes, and hence $t_* : H_n(\mathbb{R}P^n) \to H_n(S^n) \cong \mathbb{Z}_2$ is an isomorphism. Moreover, $p_* = 0$ implies that $\partial_* : H_n(\mathbb{R}P^n) \to H_{n-1}(\mathbb{R}P^n)$ is an isomorphism, and the same is true for $\partial : H_k(\mathbb{R}P^n) \to H_{k-1}(\mathbb{R}P^n)$ for $k > 0$ since $H_*(S^n) = 0$ except in degrees 0 and n. Inductively we obtain $H_k(\mathbb{R}P^n) \cong \mathbb{Z}_2$ for all $0 \leq k \leq n$.

5. Recall that $\pi_1(\mathbb{R}P^1) \cong \pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ and $\pi_1(S^n) = 0$ for $n > 1$. Hence, if $m = 1$ the only homomorphism $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2 \to \pi_1(\mathbb{R}P^1) \cong \mathbb{Z}$ is the trivial homomorphism. So from now on we may assume that we have $n > m > 1$.

$$\begin{array}{ccc} S^n & \overset{\tilde{f}}{\dashrightarrow} & S^m \\ {\scriptstyle p^n}\downarrow & & \downarrow{\scriptstyle p^m} \\ \mathbb{R}P^n & \xrightarrow{f} & \mathbb{R}P^m \end{array}$$

For any $n > m > 1$ we have

$$f_\# \circ p^n_\#(\pi_1(S^n)) = \{1\} = p^m_\#(\pi_1(S^m))$$

and $f \circ p^n : S^k \to \mathbb{R}P^m$ always lifts to a map $\tilde{f} : S^n \to S^m$.

A generator of $\pi_1(\mathbb{R}P^n)$ is represented by a loop that lifts to a path in S^n connecting two antipodal points (see also Hatcher example 1.43). The homomorphism $f_\# : \pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2 \to \pi_1(\mathbb{R}P^m) \cong \mathbb{Z}_2$ can either be an isomorphism or trivial.

$$\begin{aligned}
&f \text{ induces an isomorphism } f_\# \\
\iff &\forall \text{ path } \gamma : [0,1] \to S^n \text{ connecting antipodal points:} \\
&f_\#([p^n \circ \gamma]) = [f \circ p^n \circ \gamma] = p^m_\#[\tilde{f} \circ \gamma] \in \pi_1(\mathbb{R}P^m)\backslash\{0\} \cong \mathbb{Z}_2\backslash\{0\} \\
\iff &\forall \text{ path } \gamma : [0,1] \to S^n \text{ connecting antipodal points:} \\
&\tilde{f} \circ \gamma : [0,1] \to S^m \text{ connects antipodal points} \\
\iff &\text{the lift } \tilde{f} : S^n \to S^m \text{ is equivariant.}
\end{aligned}$$

But, since $n > m$, by Bredon Theorem 20.1 the map $\tilde{f}$ *cannot* be equivariant. Therefore, the induced map $f_\#$ must be trivial.

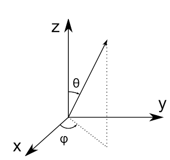
6. Assume that $r : \mathbb{R}P^3 \to \mathbb{R}P^2$ is a retraction and denote by $i : \mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$ the inclusion. Then we have $r \circ i = \mathrm{id}_{\mathbb{R}P^2}$ and hence $(r \circ i)_\# = \mathrm{id} : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2)$, which is non-zero because $\pi_1(\mathbb{R}P^2) = H_1(\mathbb{R}P^2; \mathbb{Z}) = \mathbb{Z}_2$. On the other hand, we have $(r \circ i)_\# = r_\# \circ i_\# = 0$ since $r_\# = 0$ by the previous exercise. That is a contradiction.

7. Cf. the proof of Borsuk-Ulam in [Hatcher, pp. 174-176]!

8. Let $n > m$ and supposed that there exists an equivariant map $\phi : S^n \to S^m$, i.e., such that $\phi(-x) = -\phi(x)$ for all x. Consider the map $f : S^{m+1} \to \mathbb{R}^{m+1}$ obtained by composing the restriction of ϕ to $S^{m+1} \subseteq S^n$ with the inclusion $S^m \hookrightarrow \mathbb{R}^{m+1}$. This map satisfies $f(-x) = -f(x)$ for all $x \in S^{m+1}$. Since $f(x) \in S^m$ and hence $f(x) \neq -f(x)$, we conclude $f(-x) \neq f(x)$ for all $x \in S^{m+1}$, which contradicts the Borsuk-Ulam theorem.

9. Cf. [Bredon, Corollary IV.20.4]!

10. (a) For $z \in \mathbb{R}P^k$ choose $x \in B^k_+$ such that $z = [x]$. We define $\phi(z)$ to be the point in S^k obtained from moving x down towards the South Pole S doubling the distance to the North Pole. (Explicitely for e.g. S^2, write x in spherical coordinates (φ, θ) and define $\phi(z) = (\varphi, 2\theta) \in S^k$.) ϕ: $\mathbb{R}P^k \to S^k$ descends to a homeomorphism $\mathbb{R}P^k/\mathbb{R}P^{k-1} \to S^n$.



f maps $\mathrm{Int}(B^k_\pm)$ homeomorphically onto $S^k\backslash\{S\}$.

(b) The North Pole N has two preimages under f: N and S. Near N, f is an orientation-preserving homeomorphism and hence the local degree at N is 1. Near S, f is the composition of the antipodal map with f near N. Hence the local degree at S is $(-1)^{k+1}$. We conclude $\deg(f) = 1 + (-1)^{k+1}$.
Remark. There are many choices for ϕ. One could for example also define ϕ' using $\phi'(z) = (-\varphi, 2\theta)$. Then f has local degree -1 near N and local degree $-(-1)^{k+1}$ near S. So for that choice, $\deg(f) = -(1 + (-1)^{k+1})$.
However, for any choice of ϕ as in (a), one has $\deg(f) = \pm(1 + (-1)^{k+1})$. The reason is, that f is a homeomorphism near N and a homeomorphism near S. Moreover, these

two homeomorphisms are related by the antipodal map because f factors through $\mathbb{R}P^k$. So if one of the local degrees is 1, then the other local degree will be $(-1)^{k+1}$ and if one of the local degrees is -1, then the other local degree will be $-(-1)^{k+1}$.

(c) We define a homeomorphism $g\colon \mathbb{R}P^k \cup_{h_\partial} B^{k+1} \to \mathbb{R}P^k$. g is the identity on $\mathbb{R}P^k$. To define what g does on B^{k+1}, let $j\colon B^{k+1} \to S^{k+1}$ be the inclusion of $B^{k+1} \approx B^{k+1}_+$ into S^{k+1}. Then g is defined to be $q \circ j$ on B^{k+1}. One can check, that this gives a well-defined continuous bijective map $g\colon \mathbb{R}P^k \cup_{h_\partial} B^{k+1} \to \mathbb{R}P^{k+1}$. Hence h is a homeomorphism.

(e) We compute

$$d(e^{(k+1)}) = \deg(f)e^{(k)} = (1 + (-1)^{k+1})e^{(k)} = \begin{cases} 2e^{(k)} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$