Q1 $\sigma_n : \Delta^n \longrightarrow \Delta^n/\partial\Delta^n = X$. We do by induction on $n$.

(a) Consider the pair $(\Delta^n, \partial\Delta^n)$

$$\cancel{H_n(\Delta^n)}^{0} \longrightarrow H_n(\Delta^n, \partial\Delta^n) \xrightarrow[\cong]{\partial} \tilde{H}_{n-1}(\partial\Delta^n) \longrightarrow \cancel{H_{n-1}(\Delta^n)}^{0}$$

$$H_n(\Delta^n, \partial\Delta^n) \cong H_n(X, *), \qquad \tilde{H}_{n-1}(\partial\Delta^n) \cong \mathbb{Z}$$

$\alpha_n : \Delta^n \xrightarrow{id} \Delta^n$ as an element in $H_n(\Delta^n, \partial\Delta^n)$

$\partial[\alpha_n] = \sum_{i=0}^{n} (-1)^i [F_i^n]$ , $F_i^n :=$ i-th face of $\Delta^n$ in $\tilde{H}_{n-1}(\partial\Delta^n)$

- Let $F = \bigcup_{i=1}^{n} F_i \subset \partial\Delta^n$. LES yields

$$\tilde{H}_{n-1}(\partial\Delta^n) \xrightarrow{\cong} \tilde{H}_{n-1}(\partial\Delta^n/F)$$

$$\partial[\alpha_n] \longmapsto [\sigma_{n-1}]$$

By induction hypothesis $[\sigma_{n-1}]$ generates $\tilde{H}_{n-1}(\partial\Delta^n/F)$. Thus $\partial[\alpha_n]$ generates $\tilde{H}_{n-1}(\partial\Delta^n)$.

(b) $G$ : abelian group.

$$G \longrightarrow H_n(X, * ; G) \quad g \longmapsto g\cdot\sigma_n \quad \text{isom.}$$

$n=0$: $G \to H_0(X, * ; G)$ . obvious

For arbitrary $n$, the same argument as in (a) works

Q2 $\pi : X \to Y$ : be a 2:1 covering.

$$0 \to S_\cdot(Y.\mathbb{Z}_2) \xrightarrow{T} S_\cdot(X.\mathbb{Z}_2) \xrightarrow{\pi_c} S_\cdot(Y\mathbb{Z}_2) \to 0$$

$$\rotatebox{90}{$\in$}$$

$$\sigma : \Delta^k \to Y \longmapsto \tilde{\sigma} + \theta \circ \tilde{\sigma}$$

. $\theta : X \to X$ : unique deck transformation, $\theta \neq id$.

$$\begin{tikzcd} & X \arrow[d] \\ \Delta^k \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

(a) For any $\tilde{\sigma} : \Delta^k \to X$, we have

$$T \circ \pi_c(\tilde{\sigma}) = T \circ (\pi_c \circ \tilde{\sigma}) = \tilde{\sigma} + \theta \circ \tilde{\sigma}$$

since $\tilde{\sigma}$ is a lift of $\pi_c \circ \tilde{\sigma}$. $\square$

(b) $H_i(X.\mathbb{Z}_2) \xrightarrow{\pi_{c*}} H_i(Y.\mathbb{Z}_2) \xrightarrow{T_*} H_i(X.\mathbb{Z}_2)$

By (a), $T_* \circ \pi_{c*} = id_* + \theta_*$. Since $\theta^2 = id$, $\theta^*$ cannot be trivial. The ony nontrivial endomorphism of $H_i(X.\mathbb{Z}_2) = \mathbb{Z}_2$ is the Identity. Hence $T_* \circ \pi_{c*} = id + id = 0$. $\square$

#3. $H_k(\mathbb{RP}^n; \mathbb{Z}_2) = 0$ for $k > n$. $\phi$

Consider $\pi : S^n \longrightarrow \mathbb{RP}^n$. Then

$$0 \to H_n(\mathbb{RP}^n) \xrightarrow{T} H_n(S^n) \xrightarrow{\pi_*} H_n(\mathbb{RP}^n)$$

$$\xrightarrow{\partial} H_{n-1}(\mathbb{RP}^n) \to \overset{0}{\cancel{H_{n-1}(S^n)}} \to H_{n-1}(\mathbb{RP}^n)$$

$$\xrightarrow{\partial} \cdots$$

$\leftarrow$ all in $\mathbb{Z}/2\mathbb{Z}$-coefficient

$$\xrightarrow{\partial} H_0(\mathbb{RP}^n) \xrightarrow{0} H_0(S^n) \xrightarrow{\cong} H_0(\mathbb{RP}^n) \to 0$$

$H_0(S^n) \cong \mathbb{Z}/2\mathbb{Z}$, $H_0(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$ ($\because$ $S^n$, $\mathbb{RP}^n$ are connected)

$\Rightarrow \mathbb{Z}/2\mathbb{Z} \cong H_0(\mathbb{RP}^n) \cong H_1(\mathbb{RP}^n) \cong \cdots \cong H_{n-1}(\mathbb{RP}^n)$.

<u>Claim</u> $H_n(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

Pf) We know $H_n(S^n) \cong \tilde{H}_n(S^n) = \mathbb{Z}/2\mathbb{Z}$. Since $T : H_n(\mathbb{RP}^n) \longrightarrow H_n(S^n)$ is injective, either $H_n(\mathbb{RP}^n) = 0$ or $H_n(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$. From the above LES $H_n(\mathbb{RP}^n) \neq 0$, so $H_n(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$. $\blacksquare$

#4 We first prove the following lemma:

<u>Lemma</u> Let $n > m > 1$. For any $f: \mathbb{R}P^n \to \mathbb{R}P^m$, the induced map

$$f_*: H_1(\mathbb{R}P^n; \mathbb{Z}_2) \to H_1(\mathbb{R}P^m; \mathbb{Z}_2)$$

is trivial.

Pf) Let $p_n: S^n \to \mathbb{R}P^n$, $p_m: S^m \to \mathbb{R}P^m$ be universal covers.
Take a lift $\tilde{f}: S^n \to S^m$ of $f$.

$$\begin{tikzcd} S^n \arrow[r, "\tilde{f}"] \arrow[d, "p_n"'] & S^m \arrow[d, "p_m"] \\ \mathbb{R}P^n \arrow[r, "f"] & \mathbb{R}P^m \end{tikzcd}$$

Consider the Gysin seq (everything with $\mathbb{Z}_2$-coeff).

$$\begin{tikzcd} \cdots \arrow[r] & H_k(\mathbb{R}P^n) \arrow[r, "T"] \arrow[d, "f_*"] & H_k(S^n) \arrow[r, "p_{n*}"] \arrow[d, "\tilde{f}_*"] & H_k(\mathbb{R}P^n) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{k-1}(\mathbb{R}P^n) \arrow[r] \arrow[d, "f_*"] & \cdots \\ \arrow[r] & H_k(\mathbb{R}P^m) \arrow[r, "T"] & H_k(S^m) \arrow[r] & H_k(\mathbb{R}P^m) \arrow[r, "\partial"] & H_{k-1}(\mathbb{R}P^m) \arrow[r] & {} \end{tikzcd}$$

For $k=m$, $T: H_m(\mathbb{R}P^m) \to H_k(S^m)$ is an isom. Since $H_m(S^n) = 0$,
$f_*: H_m(\mathbb{R}P^n) \to H_m(\mathbb{R}P^m)$ is trivial.
For $1 \le k \le m-1$, the boundary map $\partial: H_k(\mathbb{R}P^n) \to H_{k-1}(\mathbb{R}P^n)$ is an isom.
Therefore $f_*: H_k(\mathbb{R}P^n) \to H_k(\mathbb{R}P^m)$ is trivial for $1 \le k \le m$. ▨

Suppose $\exists$ retraction $r: \mathbb{R}P^3 \to \mathbb{R}P^2$ of $i: \mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$. Then

$$Id_*: H_1(\mathbb{R}P^2; \mathbb{Z}/2) \xrightarrow{i_*} H_1(\mathbb{R}P^3; \mathbb{Z}/2) \xrightarrow{r_*} H_1(\mathbb{R}P^2; \mathbb{Z}/2)$$

By <u>Lemma</u>, $r_*$ is trivial ↯

#5. $\phi : S^n \longrightarrow S^m$ : equiv. $\tau, \tau'$ : antipodal map. 

$$\begin{tikzcd} S^n \arrow[r, "\phi"] \arrow[loop below, "\tau"] & S^m \arrow[loop below, "\tau'"] \end{tikzcd}$$

Claim $n \leq m$

Suppose not : $n > m$. Take the standard inclusion $S^{m+1} \hookrightarrow S^n$

which is equivariant w.r.t antipodal map and consider the compositi

$$\tilde{\phi} : S^{m+1} \hookrightarrow S^n \xrightarrow{\phi} S^m \hookrightarrow \mathbb{R}^{m+1}$$

standard inclusion.

Then $\tilde{\phi}(-x) = -\tilde{\phi}(x)$. By Borsuk-Ulam, $\exists\ x \in S^{m+1}$ st

$\tilde{\phi}(-x) = \tilde{\phi}(x)$. $\lightning$ $\therefore n \leq m$.

#6. Consider the map $f : S^1 \times S^1 \longrightarrow \mathbb{R}^2$ given by

$$f : S^1 \times S^1 \xrightarrow{pr} S^1 \xhookrightarrow{i} \mathbb{R}^2$$

$pr(x.y) = x$, $i : S^1 \hookrightarrow \mathbb{R}^2$ is the standard inclusion.

If $f(x.y) = f(-x,-y)$, then $x = -x$ $\lightning$

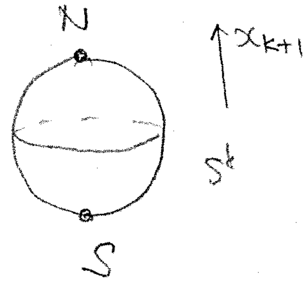
Q7 $q: S^k \to \mathbb{R}P^k$ : quotient map.

(a) Let $\partial B_+^k = \{(x_1 \cdots x_{k+1}) \in S^{k+1} : x_{k+1} \geq 0\} \subset B_+^k$.

The natural quotient map $B_+^k \to \mathbb{R}P^k \to \mathbb{R}P^k/\mathbb{R}P^{k-1}$ factors through $B_+^k/\partial B_+^k \to \mathbb{R}P^k/\mathbb{R}P^{k-1}$ which is a homeomorphism

Consider a homeomorphism $B_+^k/\partial B_+^k \to S^k$ obtained from doubling angle from the north pole. Then the composition

$$\mathbb{R}P^k/\mathbb{R}P^{k-1} \to B_+^k/\partial B_+^k \to S^k$$




satisfies the property.

(b) $f: S^k \xrightarrow{q} \mathbb{R}P^k \xrightarrow{q'} \mathbb{R}P^k/\mathbb{R}P^{k-1} \xrightarrow[\cong]{\phi} S^k$.

Let $\imath: S^k \to S^k$ be the antipodal map. We know that $\deg(\imath) = (-1)^{k+1}$ because it is a composition of $(k+1)$-reflections.

We compute $\deg(f)$ from local degrees. We have $f^{-1}(N) = \{N, S\}$.

Around N, f is orientation preserving $\Rightarrow \deg = 1$

Around S, f is given by the antipodal map $\Rightarrow \deg = (-1)^{k+1}$

$\Rightarrow \deg(f) = 1 + (-1)^{k+1}$.

Different choice of f can give overall sign $\pm$.

(c) Consider the pushout diagram

$$\begin{tikzcd}
\partial B^{k+1} = S^k \arrow[r, "h_2"] \arrow[d, hook, "j"'] & \mathbb{R}P^k \arrow[d, "r"] \arrow[ddr, bend left, "q"] & \\
B^{k+1} \arrow[r] \arrow[drr, bend right, "j"'] & \mathbb{R}P^k \cup_{h_2} B^{k+1} & \\
 & & \mathbb{R}P^{k+1}
\end{tikzcd}$$

Here $q: \mathbb{R}P^k \to \mathbb{R}^{k+1}$ is the inclusion given in the question and $7/$
$j: B^{k+1} \cong B^{k+1}_+ \hookrightarrow S^{k+1} \to \mathbb{R}P^{k+1}$ is the composition.

Since $q|_{S^k} = j|_{S^k}$, $\exists!\ g: \mathbb{R}P^k \cup_{h_2} B^{k+1} \to \mathbb{R}P^{k+1}$.

It is straightforward to check that $g$ induces a homeomorphism

$$(\mathbb{R}P^k \cup_{h_2} B^{k+1}, \mathbb{R}P^k) \cong (\mathbb{R}P^{k+1}, \mathbb{R}P^k)$$

(d). We did this in class

(e) $$C^{cw}_{k+1}(X) = \mathbb{Z}e^{(k+1)} \xrightarrow{d_{k+1}} C^{cw}_k(X) = \mathbb{Z}e^{(k)}$$

Let $f: B^{k+1} \to X$ be the characteristic map given by the composition $\phi: B^{k+1} \to \mathbb{R}P^k \cup_{h_2} B^{k+1} \xrightarrow[g]{\cong} \mathbb{R}P^{k+1} = X$

By definition,

$$d_{k+1}(e^{(k+1)}) = \deg(p \circ f|_{\partial B^{k+1}})\, \tau$$

where $\tau: B^k \to X$ is the characteristic map for $e^{(k)}$ and $p: X^{(k+1)} \to X^{(k+1)}/X^{(k)}$ is the projection

By (c) we compute $p \circ f|_{\partial B^{k+1}}$ as

$$S^k = \partial B^{k+1} \xrightarrow{h_2} \mathbb{R}P^k \to \mathbb{R}P^k/\mathbb{R}P^{k-1} \xrightarrow{\phi} S^k.$$

Hence $$\deg(p \circ f|_{\partial B^{k+1}}) = \begin{cases} 0 & k = \text{even} \\ 2 & 1 \le k \le n-1 \text{ and odd} \\ 0 & \text{o.w.} \end{cases}$$