Exercise Sheet 2

Q1 $H^1(X.\mathbb{Z})$: torsion free.

Pf) By UTC, we have

$$0 \to \mathrm{Ext}^1(H_0(X.\mathbb{Z}), \mathbb{Z}) \to H^1(X.\mathbb{Z}) \to \mathrm{Hom}(H_1(X.\mathbb{Z}), \mathbb{Z}) \to 0.$$

$H_0(X.\mathbb{Z})$ is a free $\mathbb{Z}$-module, so $\mathrm{Ext}^1(H_0(X.\mathbb{Z}), \mathbb{Z}) = 0$.

Since $\mathrm{Hom}(M.\mathbb{Z})$ is torsion-free for any $\mathbb{Z}$-mod M, $H^1(X.\mathbb{Z})$ is torsion free.

Q2. $G = \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$

(a) $T = S^1 \times S^1$

$\Rightarrow$ Cellular chain complex

$$0 \to \mathbb{Z}e^2 \xrightarrow{0} \mathbb{Z}e_1^1 \oplus \mathbb{Z}e_2^1 \xrightarrow{0} \mathbb{Z}e^0 \to 0$$

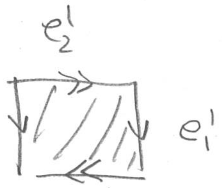
After taking $\mathrm{Hom}(-, G)$, we get

$$0 \leftarrow G \xleftarrow{0} G \oplus G \xleftarrow{0} G \leftarrow 0$$

Thus $H_i(X.\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}^2 & i=1 \\ \mathbb{Z} & i=2 \end{cases}$ $\quad H^i(X.G) = \begin{cases} G & i=0 \\ G^2 & i=1 \\ G & i=2 \end{cases}$

and UTC holds

(b) X = Klein bottle



$\Rightarrow$ Cellular chain complex

$$0 \to \mathbb{Z}e^2 \xrightarrow{(0,2)} \mathbb{Z}e_1^1 \oplus \mathbb{Z}e_2^1 \xrightarrow{0} \mathbb{Z}e^0 \to 0$$

After taking $\mathrm{Hom}(-, G)$, we get

$$0 \leftarrow G \xleftarrow{(0,2)} G \oplus G \xleftarrow{0} G \leftarrow 0$$

Q3 $f : S^n \to S^n$ $n \geq 1$ of deg d.

By UTC, we have $H^n(S^n, G) \cong \mathrm{Hom}(H_n(S^n), G)$

Since UTC is functorial, we get the following commutative diagram

$$\begin{tikzcd}
H^n(S^n, G) \arrow[r, "\cong"] \arrow[d, "f^*"'] & \mathrm{Hom}(H_n(S^n), G) \arrow[d, "(-)\circ f_*"] \arrow[r, phantom, "\cong"] & G \arrow[d, "d"] \\
H^n(S^n, G) \arrow[r, "\cong"'] & \mathrm{Hom}(H_n(S^n), G) \arrow[r, phantom, "\cong"] & G
\end{tikzcd}$$

Hence f^* is given by multiplication by d.

Q4 A : abelian group

Claim $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) \cong A_{\text{torsion}}$.

Consider $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. and associated LES:

$$\underbrace{\mathrm{Tor}(A, \mathbb{Q})}_{=0} \to \mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) \to \underbrace{\mathbb{Z} \otimes A}_{\cong A} \xrightarrow{\phi} \mathbb{Q} \otimes A \to \mathbb{Q}/\mathbb{Z} \otimes A \to 0$$

The map $\phi : A \to \mathbb{Q} \otimes_{\mathbb{Z}} A$ is given by $a \mapsto 1 \otimes a$.

It is enough to show Claim for $A = A_{\text{torsion}}$, so we assume this.

For any $a \in A$, $\exists n \in \mathbb{Z}$ s.t $na = 0$. $\Rightarrow$ $\phi = 0$.

Therefore $A_{tor} = A = \mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z})$ $\square$

Claim2 A : torsion free $\Leftrightarrow$ $\mathrm{Tor}(A, B) = 0$ $\forall B$

($\Rightarrow$) By def.

($\Leftarrow$) Take $B = \mathbb{Q}/\mathbb{Z}$ then it follows from Claim 1

Thus $H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & i=1 \\ 0 & i=2 \end{cases}$ $\quad H^i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=1 \\ \mathbb{Z}_2 & i=2 \end{cases}$

and $H^i(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & i=0 \\ \mathbb{Z}_2\oplus\mathbb{Z}_2 & i=1 \\ \mathbb{Z}_2 & i=2. \end{cases}$

UTC holds bc
$\mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G$

c) $X = \mathbb{RP}^2$.

Recall from the last exercise sheet, a cellular chain cpx is given by

$$0 \to \mathbb{Z}e^2 \xrightarrow{2} \mathbb{Z}e^1 \xrightarrow{0} \mathbb{Z}e^0 \to 0$$

After taking $\mathrm{Hom}(-, G)$, we get

$$0 \leftarrow G \xleftarrow{2} G \xleftarrow{0} G \leftarrow 0.$$

Thus,

$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}_2 & i=1 \\ 0 & i=2 \end{cases}$ $\quad H^i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=1 \\ \mathbb{Z}_2 & i=2 \end{cases}$

and $H^i(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & i=0 \\ \mathbb{Z}_2 & i=1 \\ \mathbb{Z}_2 & i=2. \end{cases}$

UTC holds

Q5. $\mathrm{Tor}(A, B)$ is always torsion.

Pf) Claim $\mathbb{Q}$ is flat over $\mathbb{Z}$. Hence $- \otimes_{\mathbb{Z}} \mathbb{Q} : \mathbb{Z}\text{-Mod} \to \mathbb{Q}\text{-Mod}$ is exact

pf) Enough to show that for any ideal $I \subset \mathbb{Z}$, $M = \mathbb{Q}$ as $\mathbb{Z}$-module $I \otimes M \to M$ is injective. For any element in $I \otimes M$, we can write it as $\alpha \otimes x$, $\alpha \in I$, $x \in M$.

$\alpha x = 0$ in $M = \mathbb{Q}$ iff $\alpha = 0$ or $x = 0$. $\square$

Now by Claim, $\mathrm{Tor}_{\mathbb{Z}}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{Tor}_{\mathbb{Q}}(A \otimes \mathbb{Q}, B \otimes \mathbb{Q})$
$= 0$

Hence $\mathrm{Tor}_{\mathbb{Z}}(A, B)$ is torsion $\boxtimes$

Q6. $X = \mathbb{R}P^2$. $G = \mathbb{Z}_2$. UTC in homology reads

$$0 \to H_i(X, \mathbb{Z}) \otimes G \to H_i(X, G) \to \mathrm{Tor}(H_{i-1}(X), G) \to 0$$

$$\Rightarrow H_0(X, G) \cong H_0(X, \mathbb{Z}) \otimes G \cong \mathbb{Z}_2$$

$$H_1(X, G) \cong (H_1(X, \mathbb{Z}) \otimes \mathbb{Z}_2) \oplus \mathrm{Tor}(H_0(X), G) \cong \mathbb{Z}_2 \oplus 0$$

$$H_2(X, G) \cong (H_2(X, \mathbb{Z}) \otimes \mathbb{Z}_2) \oplus \mathrm{Tor}(H_1(X), G) = 0 \oplus \mathbb{Z}_2$$