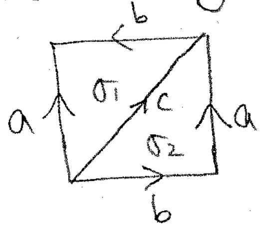
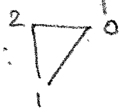
Exercise Sheet 3

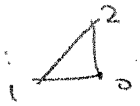
Q1 K: Klein bottle

$G = \mathbb{Z}$. We showed that $H^1(K,\mathbb{Z}) \cong \mathbb{Z}$, $H^2(K,\mathbb{Z}) \cong \mathbb{Z}/2$.

Consider the following cellular decomposition of K;



σ_1 : 

σ_2 : 

$\partial(\sigma_1) = a + b - c$

$\partial(\sigma_2) = c + a - b$.

Since $(b^* + c^*)$ is a generator of $H^1(K,\mathbb{Z}) \cong \mathbb{Z}$, we compute $(b^*+c^*)^2$.

$$(b^*+c^*)^2(\sigma_1) = (b^*+c^*)(a) \cdot (b^*+c^*)(b) = 0 \cdot 1 = 0$$

$$(b^*+c^*)^2(\sigma_2) = (b^*+c^*)(c) \cdot (b^*+c^*)(a) = 1 \cdot 0 = 0$$

$\Rightarrow (b^*+c^*)^2 = 0$. Hence, we have

$$H^*(K,\mathbb{Z}) \cong \mathbb{Z}[x,y] = \mathbb{Z}[x,y]/(x^2, y^2, xy, 2y) \quad \deg(x) = 1 \quad \deg(y) = 2$$

$G = \mathbb{Z}_2$: we showed $H^1(K,\mathbb{Z}_2) = \mathbb{Z}_2\langle a^*+c^*, b^*+c^*\rangle$ and $H^2(K,\mathbb{Z}_2) = \mathbb{Z}_2\langle \sigma_1^* = \sigma_2^* \rangle$

Only non-trivial products among a^*, b^*, c^* are

$$a^* \cup b^* = \sigma_2^* \qquad b^* \cup c^* = \sigma_1^*$$

Hence we have

$$H^*(K,\mathbb{Z}_2) \cong \mathbb{Z}_2[x,y]/(x^3, y^3, xy, x^2 - y^2) \quad \deg(x) = \deg(y) = 1$$

Q1.2/ M: oriented surface of genus g. $X = K \# M$.

X can be obtained by $(4g+4)$-gon by gluing the first 4 edges as for K and the last 4g edges for the oriented surface of genus g.

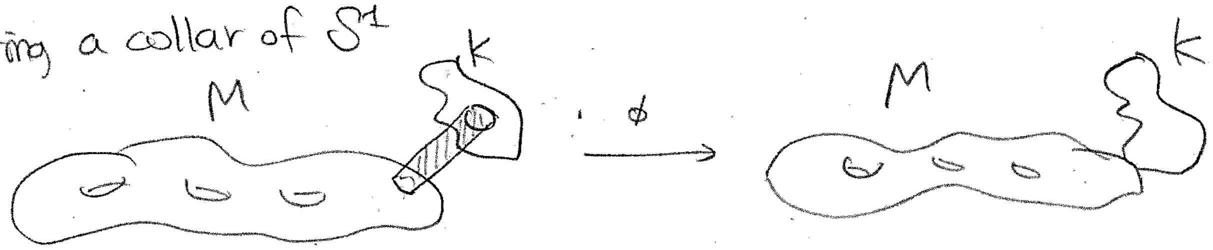
Consider the resulting cellular cochain complex

$$0 \to \underset{\deg 0}{\mathbb{Z}} \xrightarrow{0} \underset{\deg 1}{\mathbb{Z}^{2g+2}} \xrightarrow{(0,\dots,0,2)} \underset{\deg 2}{\mathbb{Z}} \to 0$$

$\Rightarrow H^0(X.\mathbb{Z}) = \mathbb{Z} \quad H^1(X.\mathbb{Z}) \cong \mathbb{Z}^{2g+1}, \ H^2(X.\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

$G = \mathbb{Z}$: Similar to Q1.1, cup product on $H^1(X.\mathbb{Z})$ is trivial

$G = \mathbb{Z}/2\mathbb{Z}$. Consider the quotient map $\phi : K\#M \longrightarrow K \vee M$ obtained by contracting a collar of S^1



We have $H^1(X.\mathbb{Z}_2) \cong (\mathbb{Z}_2)^{2g+2}$ and $H^2(X,\mathbb{Z}_2) \cong \mathbb{Z}_2$. If we denote $H_2(X.\mathbb{Z}_2) \cong \mathbb{Z}_2\langle [X] \rangle$, $H_2(K \vee M.\mathbb{Z}_2) \cong \mathbb{Z}_2\langle [M] \rangle \oplus \mathbb{Z}_2 \langle [K] \rangle$, then $\phi_*[X] = ([M], [K])$ (use cellular structure). Therefore ϕ^* induces isomorphism on $H^1(-, \mathbb{Z}_2)$ and the cup product structure of $H^1(X.\mathbb{Z}_2)$ is determined by the projection formula (see Q4).

Q2. 1/ We have $H^*(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1})$, $\deg(x)=1$.
For a proof see [Hatcher, Thm 3.19].

2/ Let $n>m$. Suppose there exists a map $f: \mathbb{RP}^n \to \mathbb{RP}^m$ inducing nontrivial map on $H^1(-;\mathbb{Z}/2)$. Let $x \in H^1(\mathbb{RP}^m; \mathbb{Z}/2)$ be the generator and $y := f^*(x) (\neq 0)$. Then by 1/

$$0 \neq y^{m+1} = (f^*(x))^{m+1} = f^*(x^{m+1}) = 0 \quad \lightning$$

Corresponding statement for $\mathbb{CP}^n$: if $n>m$, there is no map $f: \mathbb{CP}^n \to \mathbb{CP}^m$ inducing nontrivial map on $H^2(-;\mathbb{Z})$.

3/ Suppose there exist a map $f: S^n \to \mathbb{R}^n$ such that $f(x) \neq f(-x)$ for all $x \in S^n$. Let $g: S^n \to S^{n-1}$ given by

$$g(x) := \frac{f(x)-f(-x)}{|f(x)-f(-x)|}.$$

Then g satisfies $g(x) = -g(-x)$. If $n=1$, such map do not exists since S^1 is connected and g is continuous.
Let $n \geq 2$. Then g descends to a map $\bar{g}$

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & S^{n-1} \arrow[d, "\pi"] \\ \mathbb{RP}^n \arrow[r, "\bar{g}"] & \mathbb{RP}^{n-1} \end{tikzcd}.$$

Consider the Gysin sequences for $2:1$ covering $S^i \xrightarrow{\pi} \mathbb{RP}^i$ for $i=n, n-1$.

Then we get

$$\begin{tikzcd} H^0(\mathbb{RP}^{n-1};\mathbb{Z}_2) \arrow[r, "\partial", "\cong"'] \arrow[d, "\bar{g}^*"', "\cong"] & H^1(\mathbb{RP}^{n-1};\mathbb{Z}_2) \arrow[d, "\bar{g}^*"] \\ H^0(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "\partial", "\cong"'] & H^1(\mathbb{RP}^n;\mathbb{Z}_2) \end{tikzcd}$$

This shows that $\bar{g}^*$ on $H^1(-;\mathbb{Z}_2)$ should be an isomorphism, contradicting to 2/.

Q3. $X = A \cup B$, $A . B$: acyclic open.

For $k . l \geq 1$, we have the following commutative diagram

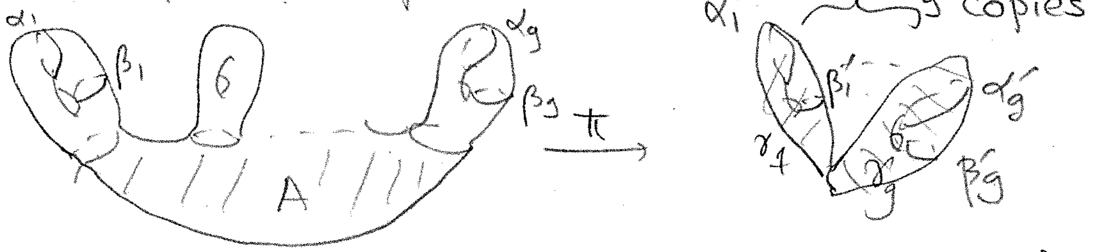
$$\begin{tikzcd} H^k(X, A; R) \times H^l(X, B; R) \arrow[r, "\cup"] \arrow[d, "\cong"'] & H^{k+l}(X, A \cup B; R) \arrow[d] \\ H^k(X; R) \times H^l(X; R) \arrow[r, "\cup"] & H^{k+l}(X; R) \end{tikzcd}$$

Left vertical map is an isomorphism because $A . B$ are acyclic.

Since $X = A \cup B$, the top horizontal map is trivial and hence the bottom map is trivial

Generalization to $X = \bigcup_{i=1}^{n} A_i$ are exactly the same.

Q4. Consider the quotient map $\pi : \Sigma_g \longrightarrow \Sigma_g / A \cong \bigvee_g T^2 =: X$



It is enough to specify cup product structure on $H^1(\Sigma_g, \mathbb{Z})$.

π^* induces an isomorphism (as $\mathbb{Q}$-vector space) on $H^1(-, \mathbb{Z})$

Let $[\Sigma_g] \in H_2(\Sigma_g, \mathbb{Z})$ be the generator. Then, we have

$$\pi_*([\Sigma_g]) = (\gamma_1, \cdots, \gamma_g) \in H_2(X) \cong \bigoplus_{i=1}^{g} H_2(T_i^2, \mathbb{Z})$$

where γ_i is a generator of $H_2(T^2, \mathbb{Z})$. Then

$$\begin{aligned}(\alpha_i \cup \beta_i)([\Sigma_g]) &= \pi^*(\alpha_i' \cup \beta_i')([\Sigma_g]) \\ &= (\alpha_i' \cup \beta_i')(\pi_*[\Sigma_g]) \\ &= (\alpha_i' \cup \beta_i')(\gamma_i) = 1\end{aligned}$$

Since π^* is a ring homomorphism, $\alpha_i \cup \alpha_j = 0$, $\beta_i \cup \beta_j = 0$

$\alpha_i \cup \beta_j = 0$ if $i \neq j$.

This specify ring structure of $H^*(\Sigma_g, \mathbb{Z})$. $\square$

Q5. Suppose $\mathbb{R}P^3$ is homotopic to $\mathbb{R}P^2 \vee S^3$. Then there exists an isomorphism of graded rings

$$H^*(\mathbb{R}P^3; \mathbb{Z}_2) \cong H^*(\mathbb{R}P^2 \vee S^3 ; \mathbb{Z}_2)$$

We compute

$$H^*(\mathbb{R}P^3, \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^4) \quad . \quad \deg(x) = 1$$

$$H^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha . \beta]/(\alpha^3, \alpha\beta, \beta^2) \quad \deg(\alpha) = 1. \ \deg(\beta) = 3.$$

Any element $\varepsilon \in H^1(\mathbb{R}P^2 \vee S^3 ; \mathbb{Z}_2)$ should satisfy $\varepsilon^3 = 0$, contradiction.