Exercise Sheet 4 (Solution)

Algebraic Topology II

15.05.2023

Q1 Let M, N be topological manifolds. Show that $M \times N$ is orientable if and only if M and N are both orientable.

Proof. We may that M, N are connected with $m = \dim M, n = \dim N$. Denote by $\widetilde{M} \to M, \widetilde{N} \to N, \widetilde{M \times N} \to M \times N$ be the orientable two-sheeted coverings. We construct first a map $\widetilde{M} \times \widetilde{N} \to \widetilde{M \times N}$. For $(x, y) \in X \times Y$ and small ball shaped neighborhoods U and V of x and y, there are maps

$$H_m(M|x) \times H_n(N|y) \to H_{m+n}(M \times N, (M \setminus x) \times (N \setminus y)) \to H_{m+n}(M \times N|(x, y))$$

defined using the cross product and the universal coefficient theorem. By naturality of cup product and the universal coefficient theorem this induces the desired continuous map $\widetilde{M} \times \widetilde{N} \to \widetilde{M \times N}$. We can now easily check that is locally two-to-one and acts on the sheets as described before.

Using the above description, the result easily follows: If both $\widetilde{M}$ and $\widetilde{N}$ are disconnected, then $\widetilde{M} \times \widetilde{N}$ has four connected components and so $\widetilde{M \times N}$ is still disconnected. Otherwise $\widetilde{M} \times \widetilde{N}$ has at most two connected components and these are glued together by the map to $\widetilde{M \times N}$. □

Q2 Show that every covering space of an orientable manifold is an orientable manifold.

Proof. Let M be an n-dimensional orientable manifold and $p : N \to M$ be a covering map. We may assume that M, N are connected. Then we have two-sheeted orientable covers $q : \widetilde{M} \to M$ and $\widetilde{N} \to N$. For any $x \in N$ and a small contractible neighborhoods $V \subset U$ we have the following commutative diagram

$$\begin{array}{ccccc} H_n(N|V) & \longleftarrow & H_n(N|U) & \longrightarrow & H_n(N|x) \\ \downarrow & & \downarrow \cong & & \downarrow \\ H_n(N|p(V) & \longleftarrow & H_n(M|p(U)) & \longrightarrow & H_n(M|p(x)) \end{array}$$

□

This shows that $\widetilde{N}$ is homeomorrphic to the fiber product

$$N \times_M \widetilde{M} := \{(n, m) \in N \times \widetilde{M} : p(n) = q(m)\}.$$

Since $\widetilde{M}$ is isomorphic to two disjoint union of M, $\widetilde{N}$ is also disconncted. Therefore N is orientable.

Q3 Show that for any connected closed orientable n-manifold M there is a map $f : M \to S^n$ of degree 1, i.e. it sends the fundamental class of M to the fundamental class of S^n.

Proof. For $x \in M$ choose an open ball $B \subset M$. Let $f : M \to M/(M \setminus B) \cong S^n$. Consider the following commutative diagram

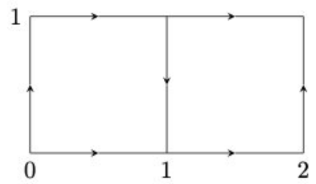
$$\begin{array}{ccccc} H_n(M) & \longrightarrow & H_n(M|x) & \longleftarrow & H_n(B|x) \\ \downarrow f_* & & \downarrow & & \downarrow \\ H_n(S^n) & \longrightarrow & H_n(M|f(x)) & \longleftarrow & H_n(f(B)|f(x)). \end{array}$$

The right vertical arrow is an isomorphisms because $f|_B : B \to f(B)$ is a homeomorphism. The two left horizontal arrows are isomorphisms by orientability and the two right horizontal arrows are isomorphisms by excision. Therefore the left vertical map is also an isomorphism and f sends a fundamental class of M to a fundamental class of S^n.

□

Q4 Find an orientable two-sheeted covering space of the Klein bottle. Which well-known space do you get?

Proof. We obtain a 2-torus T to be from the rectangle $R = [0, 2] \times [0, 1]$ by gluing the border according



to the arrows. We can also obtain a Klein bottle K by gluing the left square S. The map $R \to S$ which is the identity on the left square and $(x_1, x_2) \mapsto (x_1 - 1, 1 - x_2)$ on the right square induces a double covering $T \to K$. We want to identify this cover with the orientation double cover $\widetilde{K}$.

For this we start by choosing an orientation on R. Then we define the map $\tilde{K} \to R$ first on the image of the interior of the left square, by mapping a point $x \in K$ with a local orientation to

$$\begin{cases} (x_1, x_2), \text{if the local orientation agrees with the orientation of } R \\ (x_1 + 1, 1 - x_2), \text{ else,} \end{cases}$$

where (x_1, x_2) are the coordinates of the preimage of $x \in S$. The map is continuous and extends continuously to the image of the horizontal boundary in T. The map also extends continuously to the image of the vertical boundary since the two vertical boundaries are mapped to each other orientation reversibly and by the definition of the double cover $R \to K$. □

Q5 (1) Show that $(\alpha \cap \varphi) \cap \psi = \alpha \cap (\varphi \cup \psi)$ for all $\alpha \in C_*(X; R)$, $\varphi, \psi \in C^*(X; R)$. Deduce that the cap product makes $H_*(X; R)$ a right $H^*(X; R)$-module.
(2) Compute the module structure explicitly for X being an orientable surface of genus g and $R = \mathbb{Z}$. Do the same for X the Klein bottle.

Proof. (1) By linearity we may assume that $\alpha \in C_k(X; R), \varphi \in C^{l_1}(X; R), \psi \in C^{l_2}(X; R)$ and $k \geq l_1 + l_2$. Then we have

$$\begin{aligned}(\alpha \cap \varphi) \cap \psi = \varphi(\alpha_{[v_0 \cdots v_{l_1}]})\alpha_{[v_{l_1} \cdots v_k]} \cap \psi &= \varphi(\alpha_{[v_0 \cdots v_{l_1}]})\psi(\alpha_{[v_{l_1} \cdots v_{l_1+l_2}]})\alpha_{[v_{l_1+} \cdots v_k]} \\ &= (\varphi \cup \psi)(\alpha_{[v_0 \cdots v_{l_1+l_2}]})\alpha_{[v_{l_1+l_2} \cdots v_k]} = \alpha \cap (\varphi \cup \psi)\,.\end{aligned}$$

Therefore the cap product makes $H_*(X; R)$ a right $H^*(X; R)$-module.

(2) For the genus g surface, see Hatcher Example 3.31. For the Klein bottle, there is no homology in dimension 2, so only the cap product between H^1 and H_1 is interesting. We use the explicit simplicial complex as in Exercise Sheet 3, Q1. H_1 is generated by $a + c$ and b and H^1 is generated by $a^* + c^*$. Then $b \cap (a^* + c^*) = 0$ and $(a + c) \cap (a^* + c^*) = a \cap a^* c \cap c^* = 2[\text{pt}]$.

□