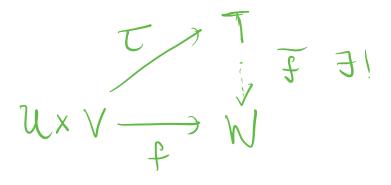
ALGEBRAIC TOPOLOGY 2

Overview:

- homology with coefficients $H_{*}(X;G)$
- cohomology : $H^{*}(X;G)$
- relation between $H_{*}(x;G) \& H_{*}(x) \otimes G^{?}$
- · algebraic operations on Hx & H*
- · Manifolds, Poincare duality
- TENSOR PRODUCT (crash course) Good references: Atiyah-McDonald Commutative Algebra Serge Lang Algebra billinear algebra -> linear algebra
- DEFINITION
- Let R be a commutative ring with unity. Let U,V be R-modules. A TENSOR PRODUCT of U & V (over R) is a R-modull T

together with a blinear map (over R) $t: \mathcal{U} \times \mathcal{V} \to \mathcal{T}$ s.t. $\forall R$ -module $W \& \forall bilinear map f: \mathcal{U} \times \mathcal{V} \to \mathcal{W}$ $\exists a$ unique homomorphism $f: \mathcal{T} \to \mathcal{W}$ s.t. $fo \mathcal{T} = f$.



LEMMA

If T exists, then it is unique up to isomorphism in the sense that if $T': UxV \rightarrow T'$ is also a tensor product of U&T then \exists : an iso. $h:T' \rightarrow T$ st $T = h \circ T'$.

Exercise.

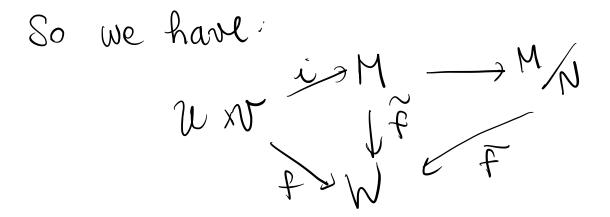
THEOREM

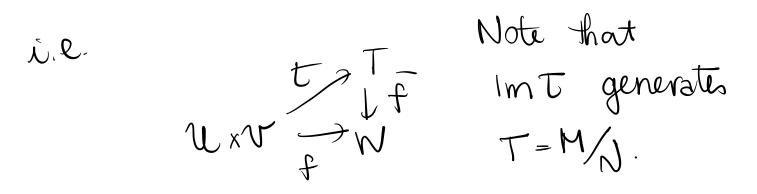
V R-moduler U&V, a tensor product of U and V exists.

Proof
Let M be the free R-module generated
by all the pairs
$$(u, v)$$
 with $u \in U_v \in V$.
Let NCM be the submodule generated
by the following elements
 $(u, v+v') - (u, v) - (u', v)$, $u, v' \in V$
 $(u, v+v') - (u, v) - (u, v')$, $v, v' \in V$
 $(u, av) - a(u, v)$
 $(u, av) - a(u, v)$

We have an injection of sets i: UXT->M. Define $t = \left(\mathcal{U}_{X} \vee \xrightarrow{i} M \longrightarrow M_{N} \right)$ Exercise: check that I is a bilinear map. Let f: UXV->W be a bilinear map. Since M'is free, we have a map $\widetilde{F}: M \rightarrow W$ s.t. the diagram U XV C J F A W

commutes. Since f is bilinear, $f|_{N} = 0$. Let $\overline{f} : M_{N} \to W$ be the map induced by \overline{f} .





(WARNING: T is generally <u>not</u> surjective) Now \widehat{F} is determined by \widehat{F} on every element in Int: As in Tgenerate T, we get that \widehat{F} is unique.

Notation: We write $U \otimes_{R} V$ for T. If R is 'clean' from the context, we write $U \otimes V$. We write $U \otimes V \coloneqq T (U, V)$, uell, vel.

YXEUQU Can be IMPORTANT: witten as $x = 2 u_i \otimes v_x$ with ui eU, VjeV. But, not VXEUDV is of the type us. ¥yell, Remark a UDV = UDaV Avev, VaeR Tensor products are hard to calculate from definitions. We often use the properties. BASIC PROPERTIES (DUOV = VOU via a unique vo. which satisfier 2(uov)=vou. $(2) (U @ V) @ W \cong U @ (V @ W)$ via an iso that satisfies $(u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w)$ (3) $U \otimes R \cong R \otimes U \cong U$.

 $\left(\begin{array}{cc} \bigoplus \mathcal{U}_i \end{array}\right) \otimes \mathcal{V} \cong \bigoplus \mathcal{U}_i \otimes \mathcal{V}$ IEI (5) If U&V are free R-modules, then so is UDT and rank (UDT) = rank 1. rank T. Furthermore, if EuijieI is a basis of U and Evij Jier is a basis for V, then Eu; OUJ JieI, JEJ is a basis of UDT. Proof 2,3 exercisés 1) We cannot just define à by putting G(uov):=vou, because it is not clean that this definition is good. Instead, define 3: VXV -> VXU bilinear

 $\mathcal{E}(u,v) = (v,u)$ Consider the following diagrams; $\mathcal{U} \times \mathcal{V} \xrightarrow{\mathcal{Z}} \mathcal{V} \times \mathcal{U} \xrightarrow{\mathcal{U}} \mathcal{V} \otimes \mathcal{U}$ t

Note that $T_{0}^{1}S_{0}^{2}$ is bilinear, so from the definition of the tensor product $\exists ! T_{0}^{2} \cdot u \oplus V \rightarrow V \oplus U$ such that the diagram above commutes. Clearly, $\exists (u \oplus v) = v \oplus u$. Exercise: Show that \exists is an isomorphism. (e.g. by constructing an inverse)

INDUCED MAPS

Let $f: \mathcal{U} \to \mathcal{U}$, $g: \mathcal{V} \to \mathcal{V}$ be R-linear maps. Then \mathcal{H} a linear map, denoted by $f \oslash g: \mathcal{U} \oslash \mathcal{V} \to \mathcal{V} \oslash \mathcal{V}$

s.t. UXV JUOV Jf×g Jføg WXVI-> NOV T'

and fog(uov)=f(u)og(v).

Exercise: Prove this. Also, $(f_1 \circ f_2) \otimes (g_1 \circ g_2) = (f_1 \otimes g_1) \circ (f_2 \otimes g_2)$. Composition of maps HOMOLOGY WITH COEFFICIENTS Fix an abelian group G. IMPORTANT: we use additive notedion for G. Neutral element is O. Let (C., 2) be a chain complex. Define a new chain complex (D.,3), by $D_i := C_i \oslash G$ S:= 2 Qid (All tensor products here are over ZZ) Exercise (D. 3) is a chain complex, ie. 303=0. Notation: D. is usually denoted by C. @G, and we write Opid for 3, but often we just write 2 again.

We can write elements of
$$C_{k} \otimes Q_{i}$$

as $\sum_{i=1}^{n} n_{i} a_{i}$ with $l \geq 0, n_{i} \in Q, a_{i} \in C_{k}$.
Invite a of $\sum_{i=1}^{n} u_{i} \otimes a_{i}$ or $\sum_{i=1}^{n} a_{i} \otimes n_{i}$.
 $\partial (\sum n_{i} a_{i}) = \sum n_{i} \geq a_{i}$
Remark: C_{k} is in general not a
part of $G \otimes C_{k}$, so \overline{A} meaning to
take $a \in C_{k}$ and consider it as
an element of $G \otimes C_{k}$.
 \overline{A} meaning to $1 \cdot a$.
Natural guestion: what is the relationship
between the old & the new chain
complex?
We'll apply the above to chain
complexes of singular chains
 $S_{i}(x), S_{i}(x, A)$.