ALGEBRAIC TOPOLOGY 2

Overview:

- homology with coefficients : $H_*(X;G)$
- cohomology : $H^*(X;G)$
- relation between $H_*(X;G)$ & $H_*(X)\otimes G$?
- algebraic operations on H_* & H^*
- Manifolds, Poincaré duality

TENSOR PRODUCT (crash course)

Good references: Atiyah-McDonald Commutative Algebra

Serge Lang Algebra

bilinear algebra $\longrightarrow$ linear algebra

DEFINITION

Let R be a commutative ring with unity.

Let $\mathcal{U},\mathcal{V}$ be R-modules. A **TENSOR PRODUCT** of $\mathcal{U}$ & $\mathcal{V}$ (over R) is a R-module T

together with a bilinear map (over R)

$\tau : U \times V \to T$ s.t. $\forall$ R-module

W & $\forall$ bilinear map $f : U \times V \to W$

$\exists$ a unique homomorphism

$\bar{f} : T \to W$ s.t. $\bar{f} \circ \tau = f$.

$$\begin{tikzcd} & T \arrow[d, dashed, "\bar{f}\ \exists!"] \\ U \times V \arrow[ur, "\tau"] \arrow[r, "f"'] & W \end{tikzcd}$$

LEMMA

If T exists, then it is unique up to isomorphism in the sense that if $\tau' : U \times V \to T'$ is also a tensor product of U & V then $\exists!$ an iso.

$h : T' \to T$ s.t. $\tau = h \circ \tau'$.

$$
\begin{tikzcd}
 & T' \arrow[d, "h\ \exists!"', "\cong"] \\
\mathcal{U} \times \mathcal{V} \arrow[ur, "t'"] \arrow[r, "t"'] & T
\end{tikzcd}
$$

Exercise.

THEOREM

$\forall$ R-modules $\mathcal{U}$ & $\mathcal{V}$, a tensor product of $\mathcal{U}$ and $\mathcal{V}$ exists.

Proof

Let M be the free R-module generated by all the pairs (u, v) with $u \in \mathcal{U}, v \in \mathcal{V}$.

Let $N \subset M$ be the submodule generated by the following elements

$$
\left\{
\begin{array}{l}
(u+u', v) - (u,v) - (u',v) \\
(u, v+v') - (u,v) - (u,v') \\
(au, v) - a(u,v) \\
(u, av) - a(u,v)
\end{array}
\right\}
\begin{array}{c}
u, u' \in \mathcal{U} \\
v, v' \in \mathcal{V} \\
a \in R
\end{array}
$$

Put $T := M/N$.

We have an injection of sets

$i : \mathcal{U} \times \mathcal{V} \to M$. Define

$$t = \left(\mathcal{U} \times V \xrightarrow{i} M \longrightarrow M/N \right)$$

Exercise: check that T is a bilinear map.

Let $f : \mathcal{U} \times \mathcal{V} \to W$ be a bilinear map.

Since M is free, we have a map

$\tilde{f} : M \to W$ s.t. the diagram

$$\begin{tikzcd}
 & M \arrow[d, "\tilde{f}"] \\
\mathcal{U} \times \mathcal{V} \arrow[ur, "i"] \arrow[dr, "f"'] \arrow[r, phantom, "\circlearrowleft"] & {} \\
 & W
\end{tikzcd}$$

commutes. Since f is bilinear,

$\tilde{f}|_N \equiv 0$. Let $\bar{f} : M/N \to W$ be the map

induced by $\tilde{f}$.

So we have:

$$\begin{tikzcd}
 & M \arrow[r] \arrow[d, "\tilde{f}"] & M/N \arrow[dl, "\bar{f}"] \\
U \times V \arrow[ur, "i"] \arrow[r, "f"'] & W &
\end{tikzcd}$$

i.e.

$$\begin{tikzcd}
 & T \arrow[d, "\bar{f}"] \\
U \times V \arrow[ur, "t"] \arrow[r, "f"'] & W
\end{tikzcd}$$

Note that $\operatorname{Im} t$ generates $T = M/N$.

(WARNING: t is generally <u>not</u> surjective)

Now $\bar{f}$ is determined by f on every element in $\operatorname{Im} t$. As $\operatorname{im} t$ generates T, we get that $\bar{f}$ is unique.

$\blacksquare$

Notation:

We write $U \otimes_R V$ for T. If R is 'clear' from the context, we write $U \otimes V$.

We write $u \otimes v := t(u,v)$, $u \in U$, $v \in V$.

IMPORTANT: $\forall x \in \mathcal{U} \otimes \mathcal{V}$ can be written as $x = \sum_{i,j} u_i \otimes v_j$ with $u_i \in \mathcal{U}, v_j \in V$. But, not $\forall x \in \mathcal{U} \otimes \mathcal{V}$ is of the type $u \otimes v$.

Remark $a\, u \otimes v = u \otimes a v$ $\quad \forall u \in \mathcal{U},\ \forall v \in \mathcal{V},\ \forall a \in \mathbb{R}$

Tensor products are hard to calculate from definitions. We often use the properties.

BASIC PROPERTIES

1. $\mathcal{U} \otimes \mathcal{V} \cong \mathcal{V} \otimes \mathcal{U}$ via a unique iso. which satisfies $\delta(u \otimes v) = v \otimes u$.

2. $(\mathcal{U} \otimes \mathcal{V}) \otimes W \cong \mathcal{U} \otimes (\mathcal{V} \otimes W)$ via an iso that satisfies
$$(u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w).$$

3. $\mathcal{U} \otimes \mathbb{R} \cong \mathbb{R} \otimes \mathcal{U} \cong \mathcal{U}$.

(4) $\left(\bigoplus_{i\in I} U_i\right)\otimes V \cong \bigoplus_{i\in I} U_i\otimes V$

(5) If U & V are free R-modules, then so is $U\otimes V$ and

$$\operatorname{rank}(U\otimes V) = \operatorname{rank} U \cdot \operatorname{rank} V.$$

Furthermore, if $\{u_i\}_{i\in I}$ is a basis of U and $\{v_j\}_{j\in J}$ is a basis for V, then $\{u_i\otimes v_j\}_{i\in I, j\in J}$ is a basis of $U\otimes V$.

Proof

(2), (3) exercises

(1) We cannot just define δ by putting $\delta(u\otimes v) := v\otimes u$, because it is not clear that this definition is good.

Instead, define $\tilde{\delta} : U\times V \to V\times U$ bilinear

$\tilde{\sigma}(u,v) = (v,u)$.

Consider the following diagrams:

$$\begin{tikzcd} U \times V \arrow[r, "\tilde{\sigma}"'] \arrow[drr, bend right, "\tau"'] & V \times U \arrow[r, "\tau'"] & V \otimes U \arrow[d, dashed, "\exists! \ \bar{\sigma}"] \\ & & U \otimes V \end{tikzcd}$$

Note that $\tau' \circ \tilde{\sigma}$ is bilinear, so from the definition of the tensor product $\exists! \ \bar{\sigma} : U \otimes V \to V \otimes U$ such that the diagram above commutes. Clearly, $\bar{\sigma}(u \otimes v) = v \otimes u$.

Exercise: Show that $\bar{\sigma}$ is an isomorphism. (e.g. by constructing an inverse)

$\blacksquare$

INDUCED MAPS

Let $f : \mathcal{U} \to \mathcal{U}'$, $g : \mathcal{V} \to \mathcal{V}'$ be R-linear maps. Then $\exists!$ a linear map, denoted by $f \otimes g : \mathcal{U} \otimes \mathcal{V} \to \mathcal{U}' \otimes \mathcal{V}'$ s.t.

$$\begin{tikzcd}
\mathcal{U} \times \mathcal{V} \arrow[r, "\tau"] \arrow[d, "f \times g"] & \mathcal{U} \otimes \mathcal{V} \arrow[d, "f \otimes g"] \\
\mathcal{U}' \times \mathcal{V}' \arrow[r, "\tau'"'] & \mathcal{U}' \otimes \mathcal{V}'
\end{tikzcd}$$

and $f \otimes g\,(u \otimes v) = f(u) \otimes g(v)$.

Exercise: Prove this.

Also, $(f_1 \circ f_2) \otimes (g_1 \circ g_2) = (f_1 \otimes g_1) \circ (f_2 \otimes g_2)$.

composition of maps

HOMOLOGY WITH COEFFICIENTS

Fix an abelian group G.

IMPORTANT: we use additive notation for G. Neutral element is 0.

Let $(C_\bullet, \partial)$ be a chain complex.

Define a new chain complex $(D_\bullet, \tilde{\partial})$, by
$$D_i := C_i \otimes G$$
$$\tilde{\partial} := \partial \otimes id$$
(All tensor products here are over $\mathbb{Z}$)

Exercise

$(D_\bullet, \tilde{\partial})$ is a chain complex, ie. $\tilde{\partial} \circ \tilde{\partial} = 0$,

Notation: $D_\bullet$ is usually denoted by $C_\bullet \otimes G$, and we write $\partial \otimes id$ for $\tilde{\partial}$, but often we just write ∂ again.

We can write elements of $C_k \otimes G$ as $\sum_{i=1}^{\ell} n_i a_i$ with $\ell \geq 0$, $n_i \in G$, $a_i \in C_k$.

instead of $\sum_{i=1}^{\ell} n_i \otimes a_i$ or $\sum_{i=1}^{\ell} a_i \otimes n_i$.

$$\partial\left(\sum n_i a_i\right) = \sum n_i \partial a_i$$

Remark: C_k is in general __not__ a part of $G \otimes C_k$, so $\nexists$ meaning to take $a \in C_k$ and consider it as an element of $G \otimes C_k$.

$\nexists$ meaning to $1 \cdot a$.

Natural question: what is the relationship between the old & the new chain complex?

We'll apply the above to chain complexes of singular chains $S_\bullet(X)$, $S_\bullet(X, A)$.