MORE ON COHOMOLOGY

Recall $H^i(X;G) := H^i(S^i(X;G))$
$= H^i(\hom(S.(X),G), \delta = \partial^*)$

$H^0(X;G)$ By UCT

$$0 \to \underbrace{\mathrm{Ext}(H_{-1}(X),G)}_{=0} \to H^0(X;G) \xrightarrow{\cong} \hom(H_0(X),G) \to 0$$

$$\Rightarrow H^0(X;G) \cong \hom\Big(\bigoplus_{c\in\pi_0(X)} \mathbb{Z}, G\Big) \cong \prod_{c\in\pi_0(X)} G$$

$H^1(X;G)$ Since $H_0(X)$ = free we have $\mathrm{Ext}(H_0(X),G)=0$, so by UCT

$$H^1(X;G) \cong \hom(H_1(X),G) \cong \hom(\pi_1(X),G)$$

every homo factors through H_1

$$\begin{tikzcd} \pi_1(X) \arrow[r, "f"] \arrow[d] & G \\ H_1(X)=\pi_1(X)^{ab} \arrow[ur, dotted, "\overline{f}"'] & \end{tikzcd}$$

b.c. G is abelian & $H_1(X)=\pi_1(X)^{ab}$

under the assumption that X is path-connected & non-empty

REDUCED COHOMOLOGY GROUPS

Consider the augmented complex

$$\cdots \to S_1(X) \to S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$\underset{\sigma}{\text{simplex}} \mapsto 1$$

To define $\tilde{H}^n(X;G)$ we take $\hom(-,G)$:

$$\tilde{H}^n(X;G) = \begin{cases} H^n(X;G) & n>0 \\ \hom(\tilde{H}_0(X),G) & n=0 \end{cases}$$

Exercise (follows from UCT applied to the augmented chain complex).

RELATIVE COHOMOLOGY GROUPS

Let X be a space, $A \subset X$ subspace, G group. We have a SES

$$0 \to S_\cdot(A) \to S_\cdot(X) \to S_\cdot(X,A) \to 0$$

CLAIM

For every degree k, this sequence splits as a SES of free abelian groups

Proof

$S_k(X,A)$ is free abelian. One can take a basis for this group to be all the chains $\sigma : \Delta_k \to X$ s.t. $\sigma(\Delta_k) \not\subset A$, viewed as elements of $S_k(X)/S_k(A)$. (Check the details) $\blacksquare$

Since this sequence splits as a sequence of abelian groups, we still have a SES after applying $\hom(-,G)$

restriction map

$$0 \to S^\circ(X,A;G) \to S^\circ(X;G) \xrightarrow{i^*} S^\circ(A;G) \to 0$$

$$S^n(X,A;G) = \text{homo } \varphi : S_n(X) \to G$$

$$\text{s.t. } \varphi|_{S_n(A)} \equiv 0$$

In cohomology we get a LES

$$\cdots \to H^n(X,A;G) \to H^n(X;G) \to H^n(A;G) \to H^{n+1}(X;A;G) \to$$

Exercise

$$\begin{tikzcd}
H^n(A;G) \arrow[r, "\delta^*"] \arrow[d, "h"] \arrow[dr, phantom, "\text{©}"] & H^{n+1}(X,A;G) \arrow[d, "h"] \\
\hom(H_n(A),G) \arrow[r, "c"'] & \hom(H_{n+1}(X,A),G)
\end{tikzcd}$$

where c is the dual map of $H_{n+1}(X,A) \xrightarrow{\partial_*} H_n(A)$.

INDUCED MAPS

$f:(X,A) \to (Y,B) \rightsquigarrow f_c : S.(X,A) \to S.(Y,B)$

$\rightsquigarrow f_c^* : S^\cdot(Y,B;G) \to S^\cdot(X,A;G)$

cochain maps

$\rightsquigarrow f^* : H^n(Y,B;G) \to H^n(X,A;G)$

the LES of (X,A) & (Y,B) are related by

$$\begin{tikzcd}
\cdots \arrow[r, "\delta^*"] & H^n(Y,B;G) \arrow[r] \arrow[d, "f^*"] & H^n(Y;G) \arrow[r] \arrow[d, "f^*"] & H^n(B;G) \arrow[r, "\delta^*"] \arrow[d, "f^*"] & \cdots \\
\cdots \arrow[r, "\delta^*"] & H^n(X,A;G) \arrow[r] & H^n(X;G) \arrow[r] & H^n(A;G) \arrow[r, "\delta^*"] & \cdots
\end{tikzcd}$$

$(X,A) \xrightarrow{f} (Y,B) \xrightarrow{g} (Z,C) \Rightarrow$

$$(f\circ g)^* = g^* \circ f^*$$

↑ contravariant functor

FURTHER PROPERTIES

① $\exists$ a SES

$$0 \to \mathrm{Ext}(H_{n-1}(X,A),G) \to H^n(X,A;G) \to \hom(H_n(X,A),G) \to 0$$

coming from UCT. The sequence is natural w.r.t. maps $(X,A) \to (Y,B)$

We can apply UCT to $S_\circ(X,A)$ because for $\forall k$, $S_k(X,A)$ is a free abelian group.

② $\exists$ also a homological version involving $H_*(X,A)\otimes G$, $\mathrm{Tor}(H_{n-1}(X,A),G)$ & $H_n(X,A;G)$.

The sequence is split, but not canonically and in fact a splitting cannot always be

arranged to be natural w.r.t. maps
between spaces.

HOMOTOPY INVARIANCE

If $f, g : (X, A) \to (Y, B)$, and $f \simeq g$
$\Rightarrow f^* = g^* : H^n(Y, B; G) \to H^n(X, A; G)$.

Proof

$\exists$ a chain homotopy between f_c & g_c,
i.e. a homo $H : S_n(X, A) \to S_{n+1}(Y, B)$
$\forall n$, s.t. $g_c - f_c = \partial \circ H + H \circ \partial \Rightarrow$

$$g_c^* - f_c^* = H^* \circ \partial^* + \partial^* \circ H^*.$$

So H^* gives a cochain homotopy between
f_c^* & g_c^*. $\Rightarrow f^* = g^*$.

EXCISION

$A \subset X$. We have $Z \subset A$, and assume
$\overline{Z} \subset \text{Int}(A)$. $\Rightarrow i : (X \setminus Z, A \setminus Z) \to (X, A)$
induces an iso $H^n(X, A; G) \to H^n(X \setminus Z, A \setminus Z; G)$.

Proof

Dualize the proof for homology.

2nd proof

Denote by $i_h : H_*(X\setminus Z, A\setminus Z) \longrightarrow H_*(X, A)$ the map induced by i.

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X\setminus Z, A\setminus Z); G) \arrow[r] & H^n(X\setminus Z, A\setminus Z; G) \arrow[r] & \hom(H_n(X\setminus Z, A\setminus Z), G) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u, equal, "\cong"] & \mathrm{Ext}(H_{n-1}(X,A); G) \arrow[r] \arrow[u, "i_h^{ext}", "\cong"'] & H^n(X,A;G) \arrow[r] \arrow[u, "i^*"] & \hom(H_n(X,A), G) \arrow[r] \arrow[u, "(i_h)^*", "\cong"'] & 0 \arrow[u, equal, "\cong"]
\end{tikzcd}
$$

The statement follows from the 5-lemma. ∎

MAYER-VIETORIS LES IN COHOMOLOGY

X space, $A, B \subset X$ subspaces.

Assume: $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$. Then $\exists$ a LES

$$\ldots \longrightarrow H^n(X;G) \to H^n(A;G) \oplus H^n(B;G) \to H^n(A\cap B;G) \to$$
$$\to H^{n+1}(X;G) \to \ldots \qquad (*)$$

This comes from the SES:

$$0 \to S_n(A \cap B) \to S_n(A) \oplus S_n(B) \to S_n^{A,B}(X) \to 0$$

subgroup of $S_n(X)$ generated by the chains that are either in A or in B

We'd like to dualize by $\hom(-, G)$ and get a SES of cochain complexes.

This is possible since $S_n^{A,B}(X) \subset S_n(X)$ is free, bc $S_n(X)$ is free, so the sequence splits.

After dualizing we use the fact that $S_\bullet^{A,B}(X) \to S_\bullet(X)$ induces an iso in homology, and from this we'll obtain that $S^\bullet(X;G) \to S^\bullet_{A,B}(X;G) := \hom(S_\bullet^{A,B}(X), G)$ induces an iso in cohomology. Passing to cohomology we get the sequence $(*)$.

$$0 \to S^{\bullet}_{A,B}(X,G) \to S^{\bullet}(A;G) \oplus S^{\bullet}(B;G) \to S^{\bullet}(A\cap B;G) \to 0$$

$$h \longmapsto \left(h|_{S.(A)},\ h|_{S.(B)}\right)$$

$$(f,g) \longmapsto f|_{S.(A\cap B)} - g_{S.(A\cap B)}$$

Another argument for why the dual of

$$0 \to S_n(A\cap B) \to S_n(A) \oplus S_n(B) \to S_n^{A,B}(X) \to 0$$

is exact from the right.

We need to show that if $\varphi : S_n(A\cap B) \to G$ is a cochain, then $\exists\ \phi : S_n(A) \to G$ & $\Psi : S_n(B) \to G$ s.t. $\phi|_{S_n(A\cap B)} - \Psi|_{S_n(A\cap B)} = \varphi$.

Indeed, we can extend φ to a G-valued function on the singular simplices of A

$$\phi(\sigma) := \begin{cases} \varphi(\sigma) & \sigma(\Delta) \subset A\cap B \\ 0 & \sigma(\Delta) \not\subset A\cap B \end{cases}$$

$$\sigma : \Delta^n \to A$$

Take $\Psi := 0$.

CELLULAR COHOMOLOGY

Let X be a CW-complex, G abelian group. $\rightsquigarrow C_\bullet^{cw}(X)$ cellular chain complex.

$$C^\bullet_{cw}(X;G) = \hom(C_\bullet^{cw}(X), G)$$

THEOREM

$H^*(C^\bullet_{cw}(X);G) \cong H^*(X;G)$.

(Works also for pairs.)

Proof

We'll apply UCT.

$$0 \to \operatorname{Ext}(H^{cw}_{n-1}(X), G) \to H^n(C^\bullet_{cw}(X;G)) \to \hom(H^{cw}_n(X), G) \to 0$$

$$\wr\!\wr \qquad\qquad\qquad\qquad\qquad\qquad \wr\!\wr$$

$$0 \to \operatorname{Ext}(H_{n-1}(X), G) \to H^n(X;G) \longrightarrow \hom(H_n(X), G) \to 0$$

We know that the SES in UCT split.

$$\begin{aligned} H^n(C^\bullet_{cw}(X;G)) &\cong \operatorname{Ext}(H^{cw}_{n-1}(X), G) \oplus \hom(H^{cw}_n(X), G) \\ &\cong \operatorname{Ext}(H_{n-1}(X), G) \oplus \hom(H_n(X), G) \\ &\cong H^n(X;G). \end{aligned}$$

$\blacksquare$