MOTIVATION

For topological spaces X, Y, we wish to compute $H_*(X \times Y)$.

It is reasonable to expect that this is expressible in terms of $H_*(X)$ and $H_*(Y)$, since for each cycle Z, $Z \times pt$ is a cycle in $X \times Y$. More generally, for cycles Z in X and W in Y, $Z \times W$ should be a cycle in $X \times Y$.

The process of establishing a connection consists of two steps:

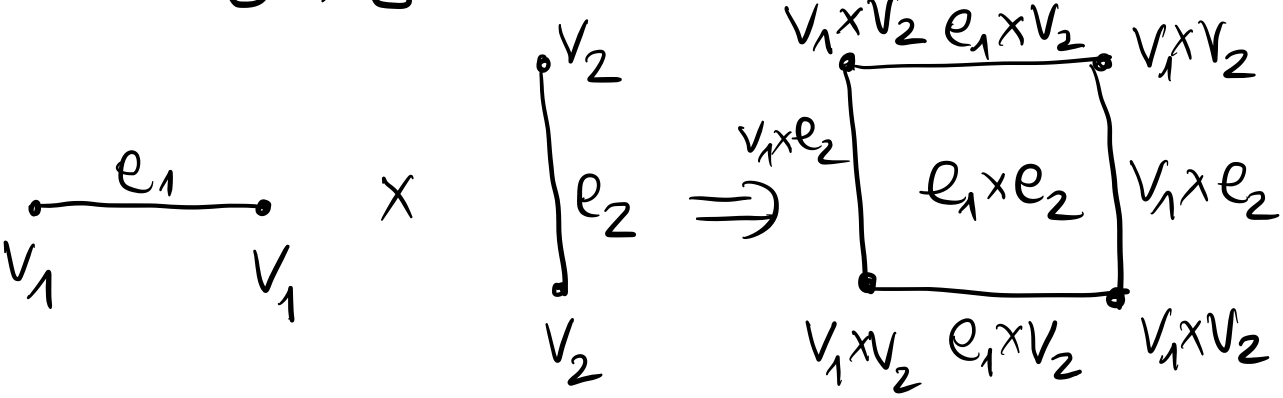
1. topology: express $S_\cdot(X \times Y)$ with $S_\cdot(X)$ and $S_\cdot(Y)$
2. algebra: compute $H_*(X \times Y)$ from $S_\cdot(X \times Y)$

What happens in step 1 is most

easily seen on an example of CW-complexes.

EXAMPLE

$T^2 = S^1 \times S^1$



CW-decomposition of T^2:

- 1 0-cell $v_1 \times v_2$
- 2 1-cells $e_1 \times v_2, v_1 \times e_2$
- 1 2-cells $e_1 \times e_2$

CW chain complex:

$$0 \to \mathbb{Z}\langle e_1 \times e_2 \rangle \xrightarrow{0} \mathbb{Z}\langle e_1 \times v_2, v_1 \times e_2 \rangle \xrightarrow{0} \mathbb{Z}\langle v_1 \times v_2 \rangle \to 0$$

product of cells of S^1 whose dim adds up to 2

product of cells of S^1 whose dim adds up to 1

For general complexes:

Let X, Y be CW-complexes, e_k a k-cell in X, e_ℓ an ℓ-cell in Y.

$\varphi : B^k \to X$ char. map for e_k

$\varphi|_{\mathring{B}^k}$ injective, $\varphi|_{S^{k-1}} =: \varphi'$ gluing map

$\Psi : B^\ell \to Y$ char. map for e_ℓ

$\Psi|_{\mathring{B}^\ell}$ injective, $\Psi|_{S^{\ell-1}} =: \Psi'$

Then $\underset{\substack{\cong \\ B^{k+\ell}}}{\varphi \times \Psi : B^k \times B^\ell} \to X \times Y$ has all

the properties of a char map:

$$\left(B^k \times B^\ell\right)^{\circ} = \mathring{B}^k \times \mathring{B}^\ell$$

$$\partial(B^k \times B^\ell) = S^{k-1} \times B^\ell \cup B^k \times S^{\ell-1}$$

$$\cong \qquad \big\downarrow \varphi' \times \Psi \cup \varphi \times \Psi'$$

$$S^{k+\ell-1} \qquad X \times Y$$

This yields a $(k+\ell)$-cell $e_k \times e_\ell$ in $X \times Y$.

Hence: n-cells in $X \times Y$ are produced by $e_k \times e_\ell$, where $k+\ell = n$.

$C_n^{cw}(X \times Y)$ is generated by $e_k \times e_\ell$, $k+\ell=n$ and we can express this algebraically as follows:

linear combinations of cells in X and in Y should give rise to linear combinations of product cells, so this correspondence must be bilinear

$$C_k^{cw} \otimes C_\ell^{cw}(Y) \to C_{k+\ell}(X \times Y)$$

$$e_k \otimes e_\ell \longmapsto e_k \times e_\ell$$

We form the tensor product of chain complexes $C_\bullet^{cw}(X)$ and $C_\bullet^{cw}(Y)$, where n-dim elements are generated by elementary tensors for which the dims of the two factors add to n

$$\left(C_\bullet^{cw}(X) \otimes C_\bullet^{cw}(Y) \right)_n = \bigoplus_{k+\ell=n} C_k^{cw}(X) \otimes C_\ell^{cw}(Y).$$

We also need to define the boundary map. this map will correspond to geometric boundary of $e_k \times e_\ell$.

For

$$B^k \times B^\ell : \partial (B^k \times B^\ell) = \partial B^k \times B^\ell \cup B^k \times \partial B^\ell$$

We can interpret the summands as chains, but we need to be careful about orientations:

$$\partial(e_k \otimes e_l) = \partial e_k \otimes e_l + \underline{(-1)^k} e_k \otimes \partial e_l$$

(to derive this formula rigorously, would require a detour to differential geometry that we will skip).

This motivates the following definitions:

GRADED GROUPS

$A_\bullet = \{A_i\}_{i \in \mathbb{Z}}$ graded abelian groups

Sometimes we write

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

$A_\bullet, B_\bullet$ are graded abelian gps. $f: A \to B$ homo. We say f is graded of degree d if $f(A_i) \subset B_{i+d}$ $\forall i$. We write $|f| = d$.

TENSOR PRODUCTS

$A_\bullet, B_\bullet$ graded abelian groups. $\Rightarrow A \otimes B$ inherits a grading.

$$(A \otimes B)_n := \bigoplus_{i+j=n} A_i \otimes B_j$$

Let $f : A_\bullet' \to B_\bullet'$, $g : A_\bullet'' \to B_\bullet''$ be graded homomorphisms.
Then $\exists$ a graded homo.

$$f \otimes g : A' \otimes A'' \longrightarrow B' \otimes B''$$

which satisfies

$$(f \otimes g)(a' \otimes a'') = (-1)^{|g| \cdot |a'|} f(a') \otimes g(a'')$$

$\forall a', a''$ elements of pure degree

Sometimes we write this as

$$\langle f \otimes g, a' \otimes a'' \rangle = (-1)^{|g||a'|} \langle f, a' \rangle \otimes \langle g, a'' \rangle$$

$|f \otimes g| = |f| + |g|$

KOSZUL SIGN CONVENTION

CHAIN COMPLEXES

$(A, \partial_A), (B, \partial_B)$ be chain complexes $\rightsquigarrow$

$(A \otimes B, \partial_{A \otimes B})$

$$\partial_{A\otimes B} := \partial_A \otimes id_B + id_A \otimes \partial_B$$

(using new sign conventions: $|\partial_A| = -1$, $|\partial_B| = -1$)

$$\partial_{A\otimes B}(a \otimes b) = (\partial_A \otimes id_B)(a \otimes b) + (id_A \otimes \partial_B)(a \otimes b)$$
$$= \partial_A(a) \otimes b + (-1)^{(-1)\cdot|a|} a \otimes \partial_B(b)$$
$$= \partial_A(a) \otimes b + (-1)^{|a|} a \otimes \partial_B(b)$$

Check that $\partial_{A\otimes B} \circ \partial_{A \otimes B} = 0$.

Special case: $(A_\bullet, \partial_A)$ chain complex

G - abelian group, viewed as a chain complex concentrated in degree 0.

$\rightsquigarrow$ $A_\bullet \otimes G$ coincides with our previous construct.

THEOREM (EILENBERG-ZILBER)

$\exists$ a chain homotopy equivalence

$$S_\bullet(X\times Y) \simeq S_\bullet(X)\otimes S_\bullet(Y)$$

that is natural in X & Y.

In particular, $\exists$ iso.

$$H_*(X\times Y) \cong H_*(S_\bullet(X)\otimes S_\bullet(Y)).$$

THE HOMOLOGY CROSS PRODUCT

X = space, denote 0-simplices in X by $x\in X$: $x:\Delta_0\to x\in X$.

THEOREM $\forall$ any two spaces X & Y $\exists$ a bilinear map

$$\begin{aligned} S_p(X)\times S_q(Y) &\xrightarrow{\times} S_{p+q}(X\times Y) \\ (\sigma,\tau) &\longmapsto \sigma\times\tau \end{aligned}$$

$\forall p,q\geq 0$ s.t.:

1. $\forall x\in X$ & $\tau:\Delta_q\to Y$

$x \times t : \Delta_q \simeq \Delta_0 \times \Delta_q \to X \times Y$ is

$(x \times t)(v) = (x, t(v)) \quad \forall v \in \Delta_q$

$\forall y \in Y$ & $\sigma : \Delta_p \to X$

$\sigma \times y : \Delta_p \to X \times Y$ is

$(\sigma \times y)(u) = (\sigma(u), y) \quad \forall u \in \Delta_p$

(2) The operation $\times$ is natural in X, Y,

namely if $f : X \to X'$, $g : Y \to Y'$

and $f \times g : X \times Y \to X' \times Y'$

$(x, y) \longmapsto (f(u), g(y))$

then $\forall a \in S_p(X), b \in S_q(Y)$ we have

$$(f \times g)_c (a \times b) = f_c(a) \times g_c(b)$$

$$\begin{tikzcd}
S_p(X) \times S_q(Y) \arrow[r, "\times"] \arrow[d, "f_c(-) \times g_c(-)"'] & S_{p+q}(X \times Y) \arrow[d, "(f \times g)_c"] \\
S_p(X') \times S_q(Y') \arrow[r] & S_{p+q}(X' \times Y')
\end{tikzcd}$$

③ $\partial(a\times b)=\partial a\times b+(-1)^{|a|}a\times\partial b$ $\forall a\in S_\bullet(X)$, $b\in S_\bullet(Y)$ of pure degree.

Remark

Since $\times$ is bilinear, it induces a linear map

$$S_\bullet(X)\otimes S_\bullet(Y)\to S_\bullet(X\times Y)$$

(in fact $\underset{a\otimes b}{S_p(X)\otimes S_q(Y)}\xrightarrow{\quad}\underset{a\times b}{S_{p+q}(X\times Y)}$)

We'll denote this operation also by $\times$.

If we endow $S_\bullet(X)\otimes S_\bullet(Y)$ with the differential $\partial_\otimes:=\partial_X\otimes id+id\otimes\partial_Y$ then the map $\times: S_\bullet(X)\otimes S_\bullet(Y)\to S_\bullet(X\times Y)$ is a chain map.

Indeed,

$$(\times\circ\partial_\otimes)(a\otimes b)=\times\circ(\partial a\otimes b+(-1)^{|a|}a\otimes\partial b)$$
$$=\partial a\times b+(-1)^{|a|}a\times\partial b=$$
$$=\partial(a\times b)=(\partial\circ\times)(a\otimes b)$$

Proof of the theorem

Step 1: Construction of a map $S_p(X) \times S_q(X) \to S_{p+q}(X)$

Acyclic models

it suffices to consider a very special case, namely when $X = \Delta_p$, $Y = \Delta_q$ are themselves simplices & the chains are identity maps (these are models, acyclic refers to the fact that $\Delta_p \times \Delta_q$ is contractible & therefore has trivial homology)

method of proof

Induction on $n = p+q$.

$\underline{n=0}$ So $p=0$, $q=0$

Define $x \times y := (x, y)$.

For higher n's & the case when $p=0$ or $q=0$, define $x \times t$, $\sigma \times y$ as in the statement. Exercise: Check that everything is satisfied.

Let $n \geq 1$, and assume we have already defined $\times$ for all spaces X, Y for all p, q with $0 \leq p+q < n$.

Let $0 < p$, $0 < q$ be such that $p+q = n$.

Take 1st $X = \Delta_p$, $Y = \Delta_q$. Let

$i_p : \Delta_p \to \Delta_p$, $i_q : \Delta_q \to \Delta_q$ be the id maps, viewed as singular simplices.

Consider $a := \partial i_p \times i_q + (-1)^p i_p \times \partial i_q \in S_{p+q-1}(\Delta_p \times \Delta_q)$

both defined by induction

Intuition: this should be $\partial(i_p \times i_q)$, but $i_p \times i_q$ has not yet been defined.

CLAIM: a is a cycle

Proof $\partial a = \partial\partial i_p \times i_q + (-1)^{p-1} \partial i_p \times \partial i_q$

by the induction hypothesis

$+ (-1)^p \partial i_p \times \partial i_q + (-1)^p (-1)^p i_p \times \partial\partial i_q$

$= 0$

($|\partial i_p \times i_q| = n-1$, $|i_p \times \partial i_q| = n-1$ so by induction we can apply the formula for ∂) ∎

But $\Delta_p \times \Delta_q$ is contractible, hence
$H_i(\Delta_p \times \Delta_q) = 0 \quad \forall\ i > 0.$

Note that $p+q-1 > 0$ because $p > 0$ & $q > 0$.

$\Rightarrow [a] = 0 \in H_{p+q-1}(\Delta_p \times \Delta_q)$

$\Rightarrow \exists\, c \in S_{p+q}(\Delta_p \times \Delta_q)$ s.t. $a = \partial c$.

Define $i_p \times i_q := c \in S_{p+q}(\Delta_p \times \Delta_q)$.

Now let $\sigma : \Delta_p \to X$, $\tau : \Delta_q \to Y$ be singular simplices. Note that

$$\sigma = \sigma_c(i_p) \ \&\ \tau = \tau_c(i_q).$$

Put $\sigma \times \tau := (\sigma \times \tau)_c (i_p \times i_q)$.

Exercise: the last definition coincides with the previous one for the case $X = \Delta_p$, $Y = \Delta_q$, $\sigma = i_p$, $\tau = i_q$.

Step 2 Map from Step 1 satisfies 2

If $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ are maps, then

$$(f \times g)_c(\sigma \times \tau) = (f \times g)_c (\sigma \times \tau)_c (i_p \times i_q)$$

product of maps

$$= ((f \circ \sigma) \times (g \circ \tau))_c (i_p \times i_q)$$

$$= f_c(\sigma) \times g_c(\tau)$$

Step 3 Map from Step 1 satisfies ③

$\partial(a\times b) = \partial((a\times b)_c(i_p\times i_q)) =$

assume a, b are singular simplices

$= (a\times b)_c \circ \partial(i_p\times i_q)$

$= (a\times b)_c(\partial i_p\times i_q + (-1)^p i_p\times\partial i_q)$

$= a_c(\partial i_p)\times b_c(i_q) + (-1)^p a_c(i_p)\times b_c(\partial i_q)$

$= \partial a_c(i_p)\times b_c(i_q) + (-1)^p a_c(i_p)\times\partial(b(i_q))$

$= \partial a\times b + (-1)^p a\times\partial b.$

Extending this map bilinearly completes the induction.