THEOREM

$\exists$ chain map $\theta : S_\cdot(X \times Y) \to S_\cdot(X) \otimes S_\cdot(Y)$, defined $\forall$ spaces X, Y, which is natural in X & Y and s.t. in degree 0 we have:

$\forall x \in X, y \in Y$, $\theta((x,y)) = x \otimes y$.

Naturality means: $\forall$ maps $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ we have a commutative diagram

$$\begin{array}{ccc} S_\cdot(X \times Y) & \xrightarrow{\theta} & S_\cdot(X) \otimes S_\cdot(Y) \\ (f \times g)_c \downarrow & & \downarrow f_c \otimes g_c \\ S_\cdot(X' \times Y') & \xrightarrow{\theta} & S_\cdot(X') \otimes S_\cdot(Y') \end{array}$$

Additionally, $\partial_\otimes \circ \theta = \theta \partial$.

Proof of theorem (acyclic models)

To construct the map do induction on the degree.

<u>$n = 0$</u> Define $\theta(x, y) = x \otimes y$ (assumption)

Let $n \geq 1$, and suppose that θ has already been defined on $S_k(X \times Y)$ $\forall\, 0 \leq k < n$

and all spaces X, Y.

Consider $k=n$. We first define Θ for the case $X=Y=\Delta_n$ and a very specific chain d_n, where $d_n : \Delta_n \to \Delta_n \times \Delta_n$ is the diagonal map ($d_n(x) = (x,x)$).

Consider $\partial d_n \in S_{n-1}(\Delta_n \times \Delta_n)$. By induction $\Theta(\partial d_n)$ is already defined.

CLAIM $\exists a_n \in (S_\bullet(\Delta_n) \otimes S_\bullet(\Delta_n))_n$ s.t. $\Theta(\partial d_n) = \partial_\otimes a_n$.

Proof of claim

$$\partial_\otimes \Theta(\partial d_n) \underset{\substack{\uparrow \\ \text{induction (well defined)}}}{=} \Theta(\partial\partial d_n) = 0 \Rightarrow$$

$\Theta(\partial d_n)$ is a cycle of degree $n-1$ in $S_\bullet(\Delta_n) \otimes S_\bullet(\Delta_n)$. If $n \geq 2$ then as Δ^n is contractible we have

$$H_{n-1}(S_\bullet(\Delta_n) \otimes S_\bullet(\Delta_n)) = 0,$$

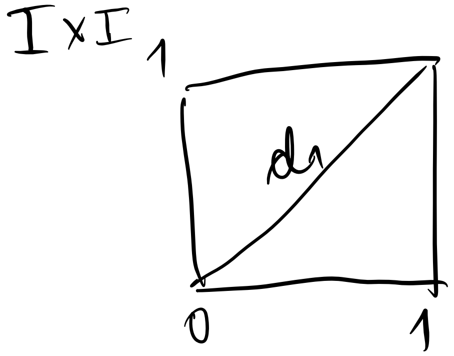
hence $\exists a_n \in (S_\bullet(\Delta_n) \otimes S_\bullet(\Delta_n))_n$ s.t.

$\theta(\partial d_n) = \partial_\otimes a_n$. This proves the claim for $n \geq 2$.

If $n = 1$, then note that

point in space

$\theta(\partial d_1) = \theta((1,1) - (0,0)) = 1 \otimes 1 - 0 \otimes 0$,



This is mapped to 0 by the augmentation map, hence its homology class is 0.

So again by the previous lemma $[\theta\partial(d_1)] =$

so $\exists a_1 \in (S(\Delta_1) \otimes S(\Delta_1))_1$ s.t. $\partial_\otimes a_1 = \theta(\partial d_1)$.

This proves the claim. $\blacksquare$

Define $\theta(d_n) = a_n$. By construction

$\partial_\otimes \theta(d_n) = \partial_\otimes a_n = \theta(\partial d_n)$.

Let now X, Y be spaces and consider

$\sigma : \Delta_n \to X \times Y$ an n-singular simplex.

Consider $(\pi_X \circ \sigma) \times (\pi_Y \circ \sigma) : \Delta_n \times \Delta_n \to X \times Y$.

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$ projections

Clearly, $\sigma = ((\pi_X \circ \sigma) \times (\pi_Y \circ \sigma)) \circ d_n : \Delta_n \to X \times Y$ as maps.

And $\sigma = ((\pi_X \circ \sigma) \times (\pi_Y \circ \sigma))_c (d_n)$
as chains

Define $\theta(\sigma) := ((\pi_X \circ \sigma)_c \otimes (\pi_Y \circ \sigma)_c)(\theta(d_n))$

Note that this is the only way to define θ (once $\theta(d_n)$ has already been defined) because

$$\begin{tikzcd} S_n(\Delta_n \times \Delta_n) \arrow[r, "((\pi_X \circ \sigma) \times (\pi_Y \circ \sigma))_c"] \arrow[d, "\theta"'] & S_n(X \times Y) \arrow[d, "\theta"] \\ (S(\Delta_n) \otimes S(\Delta_n))_n \arrow[r, "(\pi_X \circ \sigma)_c \otimes (\pi_Y \circ \sigma)_c"'] & (S(X) \otimes S(Y))_n \end{tikzcd}$$

EXERCISE Check that in case $X=\Delta_n$, $Y=\Delta_n$, $\sigma=d_n$, the new definition coincides with the previous one.

EXERCISE Check that Θ as defined above satisfies the naturality condition for maps $X \longrightarrow X'$, $Y \longrightarrow Y'$ in degrees $\leq n$.

Finally, we show that $\partial_\otimes \Theta(\sigma) = \Theta(\partial\sigma)$ $\forall \sigma \in S_n(X\times Y)$.

$$\partial_\otimes \Theta(\sigma) = \partial_\otimes\big((\pi_X\circ\sigma)_c \otimes (\pi_Y\circ\sigma)_c\big)(\Theta(d_n))$$

$$= \big((\pi_X\circ\sigma)_c \otimes (\pi_Y\circ\sigma)_c\big)\partial_\otimes\Theta(d_n) =$$

$\partial_\otimes\Theta(d_n) = \Theta\partial(d_n)$

$$= \big((\pi_X\circ\sigma)_c \otimes (\pi_Y\circ\sigma)_c\big)(\Theta(\partial d_n))$$

naturality

$$= \Theta\big(((\pi_X\circ\sigma)\times(\pi_Y\circ\sigma))_c(\partial d_n)\big)$$

$$= \Theta\circ\partial\big(((\pi_X\circ\sigma)\times(\pi_Y\circ\sigma))_c(d_n)\big)$$

$$= \Theta\partial(\sigma)$$

This completes the induction & the proof of this statement. $\blacksquare$

Our construction of x and θ involved noncanonical choices (of chains whose boundary was a given chain) so we must show that, up to chain homotopy, the particular choice made is irrelevant.

THEOREM

Let ϕ, Ψ be two chain maps, either

$$S(X \times Y) \longrightarrow S(X \times Y), \text{ or}$$

$S(X) \otimes S(Y) \longrightarrow S(X \times Y)$ or $S(X \times Y) \rightarrow S(X) \otimes S(Y)$ or $S(X) \otimes S(Y) \rightarrow S(X) \otimes S(Y)$, defined for all spaces X, Y and s.t. ϕ & Ψ are natural w.r.t. maps between spaces and s.t. ϕ & Ψ are the canonical maps in degree 0. Then $\exists$ a chain homotopy $D_{X,Y}$ between ϕ & Ψ. Moreover, we can make the chain homotopy $D_{X,Y}$ to be natural w.r. to map between spaces $X \rightarrow X'$, $Y \rightarrow Y'$.

We'll prove the version for

$$\phi, \Psi : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y).$$

We'll define $D : S_\bullet(X \times Y) \to (S_\bullet(X) \otimes S_\bullet(Y))[1]$

s.t. $D \circ \partial + \partial_\otimes \circ D = \phi - \Psi$.

Proof [these proofs are quite similar, so we skip some details]

Induction.

Put $D = 0$. This works since $\phi = \Psi$ in degree 0.

Let $n \geq 1$. Assume that D has already been defined with all the above properties $\forall X, Y$ and $0 \leq k < n$. We'll define now D on $S_n(X \times Y)$. Consider

$$d_n : \Delta_n \to \Delta_n \times \Delta_n \text{ the diagonal}$$

map viewed as an n-simplex in $S_n(\Delta_n \times \Delta_n)$.

$$\partial_\otimes(\phi - \Psi - D \circ \partial)(d_n) =$$

$$= \partial_\otimes \phi(d_n) - \partial_\otimes \Psi(d_n) - \partial_\otimes D \partial(d_n) =$$

$$= \phi(\partial d_n) - \Psi(\partial d_n) - \Big(\underbrace{\phi(\partial d_n)}_{\partial_\otimes \phi(d_n)} - \overbrace{\Psi(\partial d_n)}^{\partial_\otimes \Psi(d_n)} - \underbrace{D\circ\partial(\partial d_n)}_{=0}\Big)$$

chain maps

induction on 1 degree less

$= 0$

$\Rightarrow (\phi - \Psi - D\circ\partial)(d_n) \in (S_\bullet(\Delta_n) \otimes S_\bullet(\Delta_n))_n$ is a cycle. By the lemma from the last lecture $\exists a \in (S(\Delta_n)\otimes S(\Delta_n))_{n+1}$ s.t. $\partial a = (\phi - \Psi - D\circ\partial)(d_n)$.

Define $D(d_n) := a$.

Clearly, now we have

$$(\partial D + D\partial)(d_n) = (\phi - \Psi)(d_n)$$

Now, let $\sigma : \Delta_n \to X \times Y$ be a singular n-simplex. We have

$$\sigma = ((\pi_X \circ \sigma) \times (\pi_Y \circ \sigma))_c (d_n).$$

Define $D\delta := ((\pi_X \circ \delta)_c \otimes (\pi_Y \circ \delta)_c)(D(d_n))$.

EXERCISE Finish the proof.

$\boxtimes$

COROLLARY (EILENBERG-ZILBER THM)

'The' chain map $\times : S(X) \otimes S(Y) \to S(X \times Y)$ and $\theta : S(X \times Y) \to S(X) \otimes S(Y)$ are uniquely defined up to chain homotopy by their values in degree 0 and the requirement that they are natural in X, Y. Moreover,

$$\theta \circ \times \simeq id, \quad \times \circ \theta \simeq id$$

via chain homotopies that are natural in X, Y. In particular, $\times$ & θ are chain homotopy equivalences and $\exists$ a natural iso (w.r. to X, Y)

$$H_*(X \times Y) \cong H_*(S(X) \otimes S_\bullet(Y)).$$

If G is an abelian group, then

$$H_*(X \times Y; G) \cong H_*(S(X) \otimes S(Y) \otimes G)$$

and

$$H^*(X \times Y; G) \cong H^*(\mathrm{hom}(S(X) \otimes S(Y), G)).$$

Proof

immediately follows from the previous theorem