CROSS PRODUCT IN HOMOLOGY

Observe that since $\times : S_\cdot(X) \otimes S_\cdot(Y) \to S_\cdot(X \times Y)$ is unique up to homotopy & satisfies

$$\partial(\sigma \times \tau) = \partial\sigma \times \tau + (-1)^{|\sigma|} \sigma \times \partial\tau$$

it descend to a well defined

CROSS PRODUCT

$$\times : H_p(X) \otimes H_q(Y) \to H_{p+q}(X \times Y).$$

$$[\sigma] \times [\tau] := [\sigma \times \tau]$$

This definition is independent of choices

(If σ' is another choice with $\sigma - \sigma' = \partial\mu$, we have $\sigma \times \tau - \sigma' \times \tau = (\partial\mu) \times \tau = \partial(\mu \times \tau)$ since $\partial\tau = 0$.)

If we identify $H_*(X \times Y)$ with $H_*(Y \times X)$ via $T(x,y) = (y,x)$, then the cross product is graded commutative, ie.

$$a \times b = (-1)^{|a||b|} \, b \times a \in H_*(X \times Y)$$

Proof

We'll identify $X \times Y \approx Y \times X$ using the obvious map

$$T : X \times Y \to Y \times X$$
$$(x, y) \mapsto (y, x)$$

Since $T^2 = id$ this map induces an isomorphism

$$T_c : S_\bullet(X \times Y) \longrightarrow S_\bullet(Y \times X)$$

and similarly on homology. On the other hand consider

$$\tau : S_\bullet(X) \otimes S_\bullet(Y) \longrightarrow S_\bullet(Y) \otimes S_\bullet(X)$$
$$\alpha \otimes \beta \longmapsto (-1)^{|\alpha||\beta|} \beta \otimes \alpha$$

(the sign is required in order for τ to be a chain map (so that $\partial_\otimes \tau = \tau \partial_\otimes$). It also satisfies $\tau^2 = id$ and thus is also an isomorphism.

Consider the diagram:

$$
\begin{tikzcd}
S_\bullet(X \times Y) \arrow[r, shift left, "\theta_{X,Y}"] \arrow[d, "T_c"] & S_\bullet(X) \otimes S_\bullet(Y) \arrow[l, shift left, "\times"] \arrow[d, "\tau"] \\
S_\bullet(Y \times X) \arrow[r, shift left, "\theta_{Y,X}"] & S_\bullet(Y) \otimes S_\bullet(X) \arrow[l, shift left, "\times"]
\end{tikzcd}
$$

This diagram is not commutative, but we can show:

CLAIM

The maps $T_c^{-1} \circ \times \circ \tau$ and $\times$ are chain homotopic. Similarly $\tau^{-1} \circ \theta \circ T_c$ are chain homotopic.

The maps are natural in X & Y and are the obvious ones

$$x_0 \otimes y_0 \longmapsto x_0 \times y_0$$

$$x_0 \times y_0 \longmapsto x_0 \otimes y_0$$

in degree 0.

Conclusion: By a previous theorem there exists a chain homotopy. ∎

$\times$ is also natural with respect to maps; if $f: X \to X'$ and $g: Y \to Y'$ are continuous, then

$$(f_* a) \times (g_* b) = (f \times g)_* (a \times b) \in H_*(X' \times Y')$$

where $f \times g : X \times Y \to X' \times Y'$ is the product map. A relative version of $\times$ also exists.

Summing over p & q such that $p+q=n$ we can consider the total map

$$\times : \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \times Y)$$

and ask whether it is an isomorphism. It is always injective, but not necessarily surjective, as quantified by the Künneth formula. We give a completely algebraic version first.

THE ALGEBRAIC KÜNNETH FORMULA

Let $K_\bullet$ and $L_\bullet$ be chain complexes of free abelian groups. Then $\exists$ an exact sequence:

$$0 \to \bigoplus_{p+q=n} H_p(K_\bullet) \otimes H_q(L_\bullet) \xrightarrow{h} H_n(K_\bullet \otimes L_\bullet) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(K_\bullet), H_q(L_\bullet)) \to 0$$

This SES is natural w.r.t. chain maps $K_\bullet \to K_\bullet'$, $L_\bullet \to L_\bullet'$. The sequence splits but not canonically.

PROOF

Step 1

Consider the case that $L_\bullet$ has trivial differential so that $H_*(L_\bullet) = L_\bullet$ and $\partial_\otimes = \partial \otimes \mathrm{id}$ on $K_\bullet \otimes L_\bullet$. The homology groups

$$H_n(K_\bullet \otimes L_\bullet) = \frac{\ker\left(\partial \otimes id : (K_\bullet \otimes L_\bullet)_n \to (K_\bullet \otimes L_\bullet)_{n-1}\right)}{\operatorname{Im}\left(\partial \otimes id : (K_\bullet \otimes L_\bullet)_{n+1} \to (K_\bullet \otimes L_\bullet)_n\right)}$$

$\partial \otimes id$ preserves the direct sum decomposition:

$$(K_\bullet \otimes L_\bullet)_n = \bigoplus_{p+q=n} K_p \otimes L_q,$$

so

$$H_n(K_\bullet \otimes L_\bullet) = \bigoplus_{p+q=n} H_p(K_\bullet \otimes L_q)$$

$$\cong \bigoplus_{p+q=n} H_p(K_\bullet) \otimes L_q$$

UCT for homology (L_q is <u>free</u>, so $\mathrm{Tor}(L_q, -) = 0$)

$$\cong \bigoplus_{p+q=n} H_p(K_\bullet) \otimes H_q(L_\bullet)$$

since $L_\bullet$ has trivial boundary maps, $H_q(L_\bullet) = L_q$

Since Tor vanishes on free groups, this establishes the theorem in this case.

Step 2 (general case - sketch of proof)

Consider the SES $\left(\begin{matrix} B_n = \partial^L_{n+1}(L_{n+1}) \\ Z_n = \ker \partial^L_n \end{matrix} \right)$

$$0 \to Z_n \to L_n \to B_{n-1} \to 0$$

We apply $K_\bullet \otimes -$, which is exact since all groups involved are free, and obtain:

$$0 \to K_\bullet \otimes Z_\bullet \to K_\bullet \otimes L_\bullet \to K_\bullet \otimes B_\bullet[-1] \to 0$$

Here $Z_\bullet$ & $B_\bullet$ are viewed as chain complexes with 0-differential.

This generates a LES

$$\cdots \to H_{n+1}(K_\bullet \otimes B_\bullet[-1]) \xrightarrow{id \otimes i} H_n(K_\bullet \otimes Z_\bullet) \to$$

$$\to H_n(K_\bullet \otimes L_\bullet) \to H_n(K_\bullet \otimes B_\bullet[-1]) \xrightarrow{id \otimes i}$$

Since $Z_\bullet$ & $B_\bullet$ are trivial complexes, we have

$$H_n(K_\bullet \otimes Z_\bullet) = \bigoplus_{p+q=n} H_p(K_\bullet) \otimes Z_q$$

and

$$H_n(K_\bullet \otimes B_\bullet[-1]) = \bigoplus_{p+q=n} H_p(K_\bullet) \otimes B_{q-1} = \bigoplus_{p+q=n-1} H_p(K_\bullet) \otimes B_q$$

by the first part of the proof.

EXERCISE: The connecting homomorphism in the LES above is $id \otimes i$, where i is induced by inclusion $B_\bullet \subset L_\bullet$.

It follows that for each n we have a SES

$$0 \to \operatorname{coker}(id \otimes i)_{n+1} \to H_n(L_\bullet \otimes K_\bullet) \to \ker(id \otimes i) \to 0$$

We will obtain the Künneth formula once we identify these groups. (see the Kottke note)

Naturality & splitting is left as an exercise. $\blacksquare$

Combining the algebraic Künneth theorem with the Eilenberg-Zilber theorem yields.

TOPOLOGICAL VERSION OF THE KÜNNETH FORMULA

$\forall$ top. spaces X, Y, $\exists$ a SES

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0$$

The sequence is natural w.r.t. maps $X \to X'$, $Y \to Y'$. It splits, but not canonically. The 1st map is induced by the cross product.

EXERCISES

① Calculate the reduced homology of $T^n = S^1 \times \dots \times S^1$ (n-times).

② Calculate $H_k(\mathbb{R}P^2 \times \mathbb{R}P^3)$ and $H_k(\mathbb{R}P^2 \times \mathbb{R}P^3; \mathbb{Z}_2)$.

③ Calculate $H_k(X \times S^n)$ $(n \geq 2)$ in terms of $H_i(X)$.

① $T^1 = S^1$ & $H_i(S^1) = \begin{cases} \mathbb{Z} & i = 0,1 \\ 0 & \text{otherwise} \end{cases}$

Inductively define $T^{n+1} = S^1 \times T^n$.

Claim: $H_k(T^n) = \mathbb{Z}^{\binom{n}{k}}$.

This is the case for T^1.

Assume $H_k(T^n) = \mathbb{Z}^{\binom{n}{k}}$.

Now calculate

$$H_k(T^{n+1}) = H_k(S^1 \times T^n)$$

$$\cong \bigoplus_i \left(H_i(S^1) \otimes H_{k-i}(T^n)\right) \oplus \bigoplus_i \left(\mathrm{Tor}(H_i(S^1), H_{k-i-1}(T^n))\right)$$

Künneth formula

since all homology groups are free, this is 0

$$\cong \bigoplus_i \left(H_i(S^1) \otimes H_{k-i}(T^n)\right)$$

$$\cong \left(H_0(S^1) \otimes H_k(T^n)\right) \oplus \left(H_1(S^1) \otimes H_{k-1}(T^n)\right)$$

$$\cong H_k(T^n) \oplus H_{k-1}(T^n)$$

$$\cong \mathbb{Z}^{\binom{n}{k}} \oplus \mathbb{Z}^{\binom{n}{k-1}}$$
$$\cong \mathbb{Z}^{\binom{n+1}{k}}$$

② Recall

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i=0 \;\&\; i=n \text{ if } n \text{ odd} \\ \mathbb{Z}_2 & 0<i<n \text{ if } i \text{ odd} \\ 0 & 0<i<n \text{ if } i \text{ even} \\ 0 & i=n \text{ if } n \text{ even} \end{cases}$$

& $H_i(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2 \quad i = 0, 1, 2, \dots, n$.

Künneth formula gives

$$H_k(\mathbb{RP}^2 \times \mathbb{RP}^3) \cong \bigoplus_{i+j=k} H_i(\mathbb{RP}^2) \otimes H_j(\mathbb{RP}^3) \oplus$$
$$\bigoplus_{i+j=k-1} \mathrm{Tor}(H_i(\mathbb{RP}^2), H_j(\mathbb{RP}^3))$$

So $$H_0(\mathbb{RP}^2 \times \mathbb{RP}^3) \cong H_0(\mathbb{RP}^2) \otimes H_0(\mathbb{RP}^3)$$
$$\cong \mathbb{Z}$$

$$H_1(\mathbb{RP}^2 \times \mathbb{RP}^3) \cong \mathbb{Z} \otimes \mathbb{Z}_2 \oplus \mathbb{Z}_2 \otimes \mathbb{Z}$$
$$\oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$H_2(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$$

$$H_3(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z} \otimes \mathbb{Z} \oplus \operatorname{Tor}(\mathbb{Z}_2, \mathbb{Z}_2)$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}_2$$

$$H_4(\mathbb{R}P^2 \times \mathbb{R}P^3) \cong \mathbb{Z}_2 \otimes \mathbb{Z} \oplus 0 = \mathbb{Z}_2$$

$$H_5(\mathbb{R}P^2 \times \mathbb{R}P^3) = 0$$

$$H_6(\mathbb{R}P^2 \times \mathbb{R}P^3) = 0$$

The case for $\mathbb{Z}_2$ coefficients is left as an exercise.

3)

$$H_k(X \times S^n) \cong \bigoplus_{i+j=k} H_i(X) \otimes H_j(S^n) \oplus \bigoplus_{i+j=k-1} \operatorname{Tor}(H_i(X), H_j(S^n))$$

This gives $H_0(X \times S^n) \cong H_0(X)$

$$H_1(X \times S^n) \cong H_0(X) \otimes H_1(S^n) \oplus H_1(X) \otimes H_0(S^n)$$

$$\cong H_1(X)$$

For $k<n$ we get

$$H_k(X \times S^n) \cong H_k(X)$$

next

$$H_n(X \times S^n) \cong H_0(X) \oplus H_n(X)$$

For $k > n$

$$H_k(X \times S^n) \cong H_{n-k}(X) \oplus H_k(X).$$