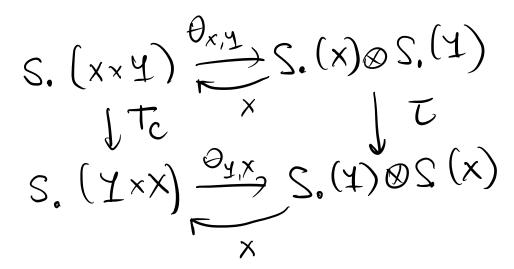
CROSS PRODUCT IN HOMOLOGY Observe that since $X : S.(X) \otimes S.(I) \rightarrow S.(X \times I)$ is unique up to homotopy & satisfies $\partial(\partial x \tau) = \partial \partial x \tau + (-1)^{10} \partial x \partial \tau$ it descend to a well defined CROSS PRODUCT $X: H_p(X) \otimes H_q(Y) \rightarrow H_{p+q}(X \times Y).$ [J × 6] =: [J] × [6] This definition is independent of choices (If 31 is another choice with 3-31=2m)

we have $\partial x t - \partial' x t = (\partial p) x t = \partial (p x t)$ Since $\partial t = 0$.

If we identify $H_{*}(x \times 1)$ with $H_{*}(Y \times X)$ Via T(x,y)=ly, x), then the cross product is graded commutative, ie. $a \times b = (-1)^{|a||b|} b \times q \in H_{\star}(x \times Y)$ Proof We'll identify $x \times Y \approx Y \times X$ using the obvious map $T: x \times Y \rightarrow Y \times X$ $(x, y) \mapsto (y, x)$

Since T²=id this map induces an isomorphism

T_c: S. $(x \times Y) \rightarrow S.(Y \times X)$ and similarly on homology. On the other hand consider $t: S.(x) \otimes S.(Y) \rightarrow S.(Y) \otimes S.(X)$ $d \otimes \beta \longrightarrow (-1)^{|d||\beta|} \beta \otimes d$ (the sign is required in order for tto be a chain map (so that $\partial_{\otimes} t = t \partial_{\otimes})$. It also satisfies $t^2 = ud$ and thus is also an isomorphism. Consider the diagram:



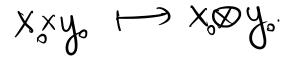
this diagram is not commutative, but we can show:

CLAIM

the maps T_c 'oxo T and X are chain homotopic. Similarly $T_o^{-1} \Theta T_c$ are chain homotopic.

the maps are natural in X&Y and are the obvious ones

 $x_{o} \oslash y_{o} \longmapsto x_{o} x y_{o}$



in degrée 0.

Conclusion: By a previous theorem there exists a chain homotopy. x is also natural with respect to maps; if $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ dre continuous, then $(f_*a)_x(g_*b) = (f_xg)_*(a_xb) \in H_*(x'xy')$ where $f \times g : X \times I \to X' \times Y'$ is the product map. A relative version of X also exists. Summing over p & g ruch that ptg-n we can consider the total map $X : \bigoplus_{p \neq g=h} H_p(x) \otimes H_g(Y) \rightarrow H_n(x \times Y)$ and ask whether it is an isomorphism. It is always injective, but not necessarily surjective, as guantified by the Künneth

formula. We give a completely algebraic version first. THE ALGEBRAIC KÜNNETH FORMULA Let K, and L. be chain complexes of free abelian groups. Then J an exact seguence : $0 \rightarrow \bigoplus H_p(K_0) \otimes H_q(L_0) \rightarrow H_n(K_0 L_0) \rightarrow \bigoplus Tor(H_p(K_0), H_q(U)) \rightarrow D$ p+g=n p+g=n-1 p+g=n-1This SES is natural w.r.t. chain maps K. -> K. , L. -> L. . The septence splits but not canonically. PROOF Step 1 Consider the case that L. has trivial differential so that $H_*(L_{\bullet}) = L_{\bullet}$ and $\partial_{\Theta} = \partial \Theta id$ on K. $\otimes L_{\bullet}$. The homology groups

$$H_{n} (K, \mathcal{O}_{L}) = \frac{keu (\partial \mathcal{O}_{i}di (K, \mathcal{O}_{L})_{n} \rightarrow (K, \mathcal{O}_{L})_{n-1})}{M (\partial \mathcal{O}_{i}di (K, \mathcal{O}_{L})_{n+1} \rightarrow (K, \mathcal{O}_{L})_{n})}$$

$$\partial \mathcal{O}_{i}d \quad \text{preserves the direct sum}$$

$$decomposition: (K, \mathcal{O}_{L},)_{n} = \bigoplus_{p+g=n} K_{p} \mathcal{O}_{L}_{g},$$
So
$$H_{n} (K, \mathcal{O}_{L},) = \bigoplus_{p+g=n} H_{p} (K, \mathcal{O}_{L}_{g})$$

$$\int_{p+g=n}^{\infty} \bigoplus_{p+g=n} H_{p} (K, \mathcal{O}_{L}_{g})$$

$$\int_{p+g=n}^{\infty} \bigoplus_{p+g=n} H_{p} (K, \mathcal{O}_{L}_{g})$$
Since L, has $\int_{p+g=n}^{\infty} \bigoplus_{p+g=n} H_{p} (K, \mathcal{O}_{L}_{g}(L,))$

$$for homology$$

$$(L_{g} \text{ is free}, \\ \text{so Tor} (L_{g}, -) = \mathcal{O})$$
Since L, has $\int_{p+g=n}^{\infty} \bigoplus_{p+g=n} H_{p} (K, \mathcal{O}_{L}_{g}(L,))$

$$H_{g} (L_{L}) = L_{g}$$

Since Tor vanishes on free groups, this establishes the theorem in this case. Step 2 (general case-sketch of proof) Consider the SES $\begin{pmatrix} B_n = \partial_{n+1}^{L}(L_{n+1}) \\ Z_n = \ker \partial_n^{L} \end{pmatrix}$ $0 \rightarrow Z_n \rightarrow L_n \rightarrow B_{n-1} \rightarrow 0$ We apply K. @ -, which is exact since all groups involved are free, and obtain: 0→K.@Z.→K.@L.→K.@B.[-1]→0 Here Z. & B. are viewed as chaim complexes with 0-differential. This generates a LES $\rightarrow H_{n+1}(K_{\bullet} \otimes B_{\bullet} E_{\cdot}) \xrightarrow{id \otimes i} H_{n}(K_{\bullet} \otimes Z_{\bullet}) \rightarrow$ $\rightarrow H_n(K_{\bullet}\otimes L_{\bullet}) \rightarrow H_n(K_{\bullet}\otimes B_{\bullet}(-1)) \xrightarrow{id_{\otimes i}}$ Since Z. & B. are trivial complexes, we have

$$H_{n}(K, \emptyset Z_{*}) = \bigoplus H_{P}(K_{*}) \emptyset Z_{2}$$

and
$$H_{n}(K_{*} \emptyset B_{*}[-1]) = \bigoplus H_{P}(K_{*}) \emptyset B_{2} = \bigoplus H_{P}(K_{*}) \emptyset B$$

Combining the algebraic Künneth theorem with the Eilenberg-Zilber theorem yields TOPOLOGICAL VERSION OF THE KÜNNETH FORMULA ¥ top. spaces X, I, Ja SES $0 \rightarrow \oplus H_p(x) \otimes H_g(x) \rightarrow H_n(xxx) \rightarrow \oplus \text{tor}(H_p(x)H(y)) \rightarrow 0$ p+g=n p+g=n-i p+g=n-iThe sequence is notural w.r.t. maps $X \rightarrow X', Y \rightarrow Y'$. It splits, but not canonically. The 1st map is induced by the cross product. EXERCISES 1) Calculate the reduced homology of $+n = S_{X-X} S_1 (n-times).$ 2 Calculate H_k(RP² x RP³) and $H_{K}(\mathbb{RP}^{2} \times \mathbb{RP}^{3}; \mathbb{Z}_{2}).$

(3) Calculate
$$H_{k}(x,x^{n})(n \ge 2)$$
 in terms
of $H_{i}(x)$.
(1) $T^{1}=S^{1} \otimes H_{i}(S^{1})= \begin{cases} Z & i=0,1 \\ 0 & Otherwise \end{cases}$
Inductively define $T^{n+1}=S^{1}\times T^{n}$.
Claim: $H_{k}(T^{n})=Z^{\binom{n}{k}}$.
This is the case for T^{1} .
Assume $H_{k}(T^{n})=Z^{\binom{n}{k}}$.
Now calculate
 $H_{k}(T^{n+1})=H_{k}(S^{1}\times T^{n})$
 $\stackrel{\cong}{=} \bigoplus_{i}(H_{i}(S^{i})\otimes H_{k-i}(T^{n})) \xrightarrow{\text{Nondossy opouts}}_{i} formula$
 $\stackrel{\cong}{=} \bigoplus_{i}(H_{i}(S^{i})\otimes H_{k-i}(T^{n})) \xrightarrow{\text{Condossy opouts}}_{i} formula$
 $\stackrel{\cong}{=} \bigoplus_{i}(H_{i}(S^{i})\otimes H_{k-i}(T^{n})) \xrightarrow{\text{Output}}_{i} formula$
 $\stackrel{\cong}{=} \bigoplus_{i}(H_{i}(S^{i})\otimes H_{k-i}(T^{n})) \bigoplus_{i}(H_{i}(S^{i})\otimes H_{k-i}(T^{n}))$
 $\stackrel{\cong}{=} (H_{0}(S^{i})\otimes H_{k}(T^{n})) \bigoplus_{i}(H_{i}(S^{i})\otimes H_{k-i}(T^{n}))$

 $\approx 72^{(n)} \oplus 72^{(n-1)}$ $\simeq 77^{\binom{n+1}{k}}$

$$\begin{array}{c} \textcircled{2} \ \mbox{Recall} \\ H_{i}(\mathbb{R}\mathbb{P}^{n}) = \begin{cases} \mathbb{Z} & i=0 \ \mbox{i=n if n odd} \\ \mathbb{Z}_{2} & 0 < i < n \ \mbox{if i odd} \\ 0 & 0 < i < n \ \mbox{if i even} \\ 0 & i = n \ \mbox{if n even} \end{cases} \end{cases}$$

$$H_{2} (\mathbb{R}P^{2} \times \mathbb{R}P^{3}) \cong \mathbb{Z}_{2} \otimes \mathbb{Z}_{2} \cong \mathbb{Z}_{2}$$

$$H_{3} (\mathbb{R}P^{2} \times \mathbb{R}P^{3}) \cong \mathbb{Z} \otimes \mathbb{Z} \oplus \operatorname{Tor}(\mathbb{Z}_{2}, \mathbb{Z}_{2})$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}_{2}$$

$$H_{4} (\mathbb{R}P^{2} \times \mathbb{R}P^{3}) \cong \mathbb{Q}_{2} \otimes \mathbb{Z} \oplus \mathbb{Q} = \mathbb{Z}_{2}$$

$$H_{5} (\mathbb{R}P^{2} \times \mathbb{R}P^{3}) = \mathbb{Q}$$

$$H_{6} (\mathbb{R}P^{2} \times \mathbb{R}P^{3}) = \mathbb{Q}$$
The case for $\mathbb{Z}_{2} \mod ficients$

$$\lim_{\substack{k \in \mathbb{Q} \\ k \in$$

For k<n we get $H_{\kappa}(X \times S^{n}) = H_{\kappa}(x)$ $H_n(X \times S^n) \cong H_0(X) \oplus H_n(X).$ next For Km $H_{\kappa}(x \times S^n) \cong H_{n-\kappa}(x) \oplus H_{\kappa}(x)$.