COHOMOLOGICAL CROSS PRODUCT

Fix a commutative ring R (with a unity). Let X, Y be spaces. Let $\varphi \in S^p(X;R)$, $\Psi \in S^q(Y;R)$ be cochains.

We'll define $\varphi \times \Psi \in S^n(X \times Y; R)$, where $n = p+q$.

Write $\varphi : S_p(X) \to R$, $\Psi : S_q(Y) \to R$.

Recall $\theta : S_\bullet(X \times Y) \to S_\bullet(X) \otimes_{\mathbb{Z}} S_\bullet(Y)$.

Fix one such map. Consider now

$$m \circ \varphi \otimes \Psi : S_p(X) \otimes_{\mathbb{Z}} S_q(Y) \to R \otimes_{\mathbb{Z}} R \xrightarrow{m} R,$$

where the map m is induced by the bilinear map $R \times R \to R$, $(r_1, r_2) \mapsto r_1 \cdot r_2$.

$\Rightarrow$ We get a cochain

$$m \circ \varphi \otimes \Psi : \underbrace{(S_\bullet(X) \otimes S_\bullet(Y))_{p+q}} \to R$$

$$\left(\bigoplus_{p'+q'=p+q}^{\shortparallel} S_{p'}(X) \otimes_{\mathbb{Z}} S_{q'}(Y) \right)$$

by defining $m \circ \varphi \otimes \psi$ to be 0 $\forall p', q'$ s.t. $p'+q' = p+q$ but $(p', q') \neq (p, q)$.

Define

$$\varphi \times \psi := m \circ (\varphi \otimes \psi) \circ \Theta, \quad \varphi \times \psi \in S^{p+q}(X \times Y; R)$$

$(\Theta : S_\bullet(X \times Y) \to S_\bullet(X) \otimes_{\mathbb{Z}} S_\bullet(Y))$

<u>Note</u>: $\times$ is natural w.r.t. map $X \to X'$, $Y \to Y'$ because Θ has this property.

Θ is not unique.

A more explicit formula:

Let $c \in S_n(X \times Y)$, where $n = p+q$.

$$\Theta(c) = \sum_{r+s=n} \sum_{i,j} a_i^r \otimes b_j^s \quad \text{with}$$

$a_i^r \in S_r(X)$, $b_j^s \in S_s(Y)$.

$$(\varphi \times \Psi)(c) = (-1)^{p \cdot q} \sum_{i,j} \varphi(a_i^p) \cdot \Psi(b_j^q)$$

Proof

$$(\varphi \times \Psi)(c) = m \circ (\varphi \otimes \Psi) \circ \Theta(c) =$$

$$= m \circ (\varphi \otimes \Psi) \Big(\sum_{r+s=n} \sum_{i,j} a_i^r \otimes b_j^s \Big) =$$

$$= \sum_{r+s=n} \sum_{i,j} m \circ \varphi \otimes \Psi \, (a_i^r \otimes b_j^s) =$$

$$= \sum_{i,j} m \circ \varphi \otimes \Psi \, (a_i^p \otimes b_j^q) =$$

$$= \sum_{i,j} m \Big((-1)^{|\Psi| p} \varphi(a_i^p) \otimes \Psi(b_j^q) \Big)$$

$$= \sum_{i,j} (-1)^{pq} \varphi(a_i^p) \cdot \Psi(b_j^q)$$

$\blacksquare$

CLAIM

$$\delta(\varphi \times \Psi) = \delta\varphi \times \Psi + (-1)^{|\varphi|}\varphi \times \delta\Psi.$$

In other words, the map

$$S^p(X;R) \otimes_{\mathbb{Z}} S^q(Y;R) \to S^{p+q}(X \times Y;R)$$

induced by

$$(\varphi, \Psi) \mapsto \varphi \times \Psi \text{ is a chain map.}$$

(w.r.t. $\delta_X \otimes id + id \otimes \delta_Y$ and $\delta_{X \times Y}$).

WARNING:

In homological algebra, δ is often defined as: $\delta f = (-1)^{\deg f + 1} f \circ \partial$.

(Sign Covention, Bredon, page 321).

This change does not affect cohomology groups.

Proof of claim

$p := |\varphi|$

$q := |\Psi|$

$$\delta(\varphi\times\psi) = (-1)^{p+q+1}(\varphi\times\psi)\circ\partial$$

special convention

$$= (-1)^{p+q+1}\, m\circ(\varphi\otimes\psi)\circ\Theta\circ\partial$$

$$= (-1)^{p+q+1}\, m\circ(\varphi\otimes\psi)\circ\partial_{\otimes}\circ\Theta$$

$$\partial_X\otimes id + id\otimes\partial_Y$$

$$= (-1)^{p+q+1}\, m\circ\left(\varphi\otimes(\psi\circ\partial_Y) + (-1)^q(\varphi\circ\partial_X)\otimes\psi\right)\circ\Theta =$$

$$= (-1)^{p+q+1}\, m\circ\left((-1)^{q+1}\varphi\otimes\delta\psi + (-1)^{q+p+1}\delta\varphi\otimes\psi\right)\circ\Theta$$

$$= m\circ\left(\delta\varphi\otimes\psi + (-1)^p\varphi\otimes\delta\psi\right)\circ\Theta$$

$$= \delta\varphi\times\psi + (-1)^p\varphi\times\delta\psi$$

$\blacksquare$

REMARK

The map $(\varphi,\psi)\longrightarrow\varphi\times\psi$ is bilinear over R, so it induces a map of R-mod

$$S^p(X;R)\otimes_R S^q(Y;R)\to S^{p+q}(X\times Y;R).$$

Exercise.

COROLLARY

Cross product on chains induces a product

$$H^p(X;R) \otimes_R H^q(Y;R) \to H^{p+q}(X \times Y;R)$$

which is independent on the choice of Θ.

Proof

We have the following maps

$$\underbrace{H^p(S^\bullet(X;R)) \otimes_R H^q(S^\bullet(Y;R))}_{H^p(X;R) \otimes_R H^q(Y;R)} \xrightarrow{h} H^{p+q}(S^\bullet(X;R) \otimes_R S^\bullet(Y;R))$$

$\uparrow$ algebraic map $[a] \otimes [b] \to [a \otimes b]$

$$\xrightarrow{\times} \underbrace{H^{p+q}(S^\bullet(X \times Y;R))}_{H^{p+q}(X \times Y;R)}$$

also denoted $\times$

The composition $\times \circ h$ gives the desired map. The independence of the specific choice of Θ follows from the fact that Θ is unique up to chain homotopy. ∎

COMMUTATIVITY OF THE CROSS PRODUCT

Let $\alpha \in H^p(X)$, $\beta \in H^q(Y)$.

Fix a ring R for coefficients & omit from notation.

Question: What is the relation between $\alpha \times \beta \in H^{p+q}(X \times Y)$ and $\beta \times \alpha \in H^{p+q}(Y \times X)$?

We'll identify $X \times Y \cong Y \times X$ using the obvious map

$$T : X \times Y \to Y \times X$$
$$(x, y) \mapsto (y, x)$$

Consider the following diagram:

$$\begin{tikzcd} S_\bullet(X \times Y) \arrow[r, "\theta_{X,Y}"] \arrow[d, "T_C"] & S_\bullet(X) \otimes S_\bullet(Y) \\ S_\bullet(Y \times X) \arrow[r, "\theta_{Y,X}"] & S_\bullet(Y) \otimes S_\bullet(X) \arrow[u, "\tau"'] \end{tikzcd}$$

$\tau(b \otimes a) = (-1)^{|b| \cdot |a|} a \otimes b.$

Exercise. τ is a chain map.

Consider the composition

$\tau \circ \Theta_{Y,X} \circ T_C$. This is a chain map

$S.(X \times Y) \to S.(X) \otimes S.(Y)$. It is natural wrt

maps $X \to X'$, $Y \to Y'$, and in degree 0

this map does $(x,y) \mapsto x \otimes y$.

Conclusion: By a previous theorem

there exists a chain homotopy

$\tau \circ \Theta_{Y,X} \circ T_C \simeq \Theta_{X,Y}$, i.e. there is

an operator $D : S.(X \times Y) \to (S.(X) \otimes S.(Y))[1]$

s.t.

$$\tau \circ \Theta_{Y,X} \circ T_C - \Theta_{X,Y} = D \circ \partial + \partial_{\otimes} \circ D.$$

We now pass to cohomology. Let

$f \in S^p(X)$, $g \in S^q(Y)$ be cocycles.

Let's calculate

$$T^*([g] \times [f]):$$

$\times$ on the level of chomology groups

$\times$ on the level of cochains

$$T^*([g] \times [f]) = T^*([g \times f]) =$$

$$= T^*([m\circ(g\otimes f)\circ \theta_{Y,X}]) = [m\circ(g\otimes f)\circ\theta_{Y,X}\circ T_C]$$

$$= (-1)^{p\cdot q}[m\circ(f\otimes g)\circ\tau\circ\theta_{Y,X}\circ T_C] =$$

$$= (-1)^{p\cdot q}[m\circ(f\otimes g)\circ\theta_{X,Y} + m\circ(f\otimes g)\circ(D\partial + \partial_\otimes D)]$$

The 2nd term in the [...]: it is a coboundary because f & g are cocycles. So $(f\otimes g)\circ\partial_\otimes = 0$,

and $m\circ(f\otimes g)\circ D\partial = \pm\delta(m\circ(f\otimes g)\circ D)$.

$$\Rightarrow T^*([g]\times[f]) = (-1)^{pq}[f]\times[g].$$

skew / super commutativity

We proved

PROPOSITION

$\forall\ \alpha \in H^p(X;R),\ \beta \in H^q(Y;R)$ we have $\alpha \times \beta = (-1)^{p \cdot q}\, T^*(\beta \times \alpha)$.

The operation that we will use most in the remaining weeks is the cup product.