PROPOSITION

$\forall\ \alpha \in H^p(X;R),\ \beta \in H^q(Y;R)$ we have $\alpha \times \beta = (-1)^{p \cdot q}\, T^*(\beta \times \alpha)$.

The operation that we will use most in the remaining weeks is the cup product.

THE CUP PRODUCT

Let X be a space. Fix a ring of coefficients R. We'll define an operation

$$H^p(X;R) \otimes_R H^q(X;R) \xrightarrow{\cup} H^{p+q}(X;R)$$

$$\alpha \otimes \beta \longmapsto \alpha \cup \beta$$

Consider the diagonal map

$$d : X \to X \times X,\ d(x) = (x,x).$$

Define $\alpha \cup \beta := d^*(\alpha \times \beta)$.

The operation is independent of the choice of θ in cohomology.

Cup product is graded commutative.

$$\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha \quad \forall \alpha \in H^p(X;R),\ \beta \in H^q(X;R)$$

Proof of graded commutativity

$$\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha \quad \forall \alpha \in H^p(X), \beta \in H^q(X)$$

$$\beta \cup \alpha = d^* (\beta \times \alpha) = (-1)^{pq} d^* T^* (\alpha \times \beta)$$

$$T: X \times X \to X \times X$$
$$(x,y) \mapsto (y,x)$$

But $d^* \circ T^* = (T \circ d)^* = d^*$ because

$T \circ d = d$. So

$$\beta \cup \alpha = \dots = (-1)^{pq} d^* (\alpha \times \beta) = (-1)^{pq} \alpha \cup \beta$$

$\blacksquare$

We defined the cup product using the cross product.

It turns out that the cup product also determines the cross product.

Cup product on the level of cochains involves the dual of the composition

$$S_\bullet(X) \xrightarrow{d_*} S_\bullet(X \times X) \xrightarrow{\theta} S_\bullet(X) \otimes S_\bullet(X)$$

Any chain map $S_\bullet(X) \to S_\bullet(X) \otimes S_\bullet(X)$ which is natural in X and is the obvious map $x_0 \mapsto x_0 \otimes x_0$ on 0-chains is called diagonal approximation. It can be shown that any two diagonal approximations are chain homotopic. Taking a diagonal approximation with front & back faces of simplices (Alexander-Whitney diagonal approximation) leads to a more concrete definition of the cup product as presented in Hatcher.

THE CUP PRODUCT MORE CONCRETELY

Let $0 \leq k \leq n$. Regard the standard

n-simplex $\Delta_n = [e_0, \dots, e_n]$ as a map id_{Δ_n} (so as a singular simplex of the space Δ_n).

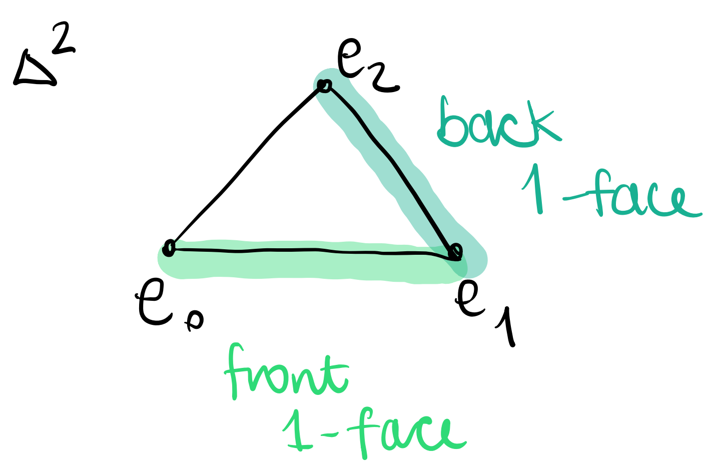
The front k-face of Δ^n is the singular k-simplex

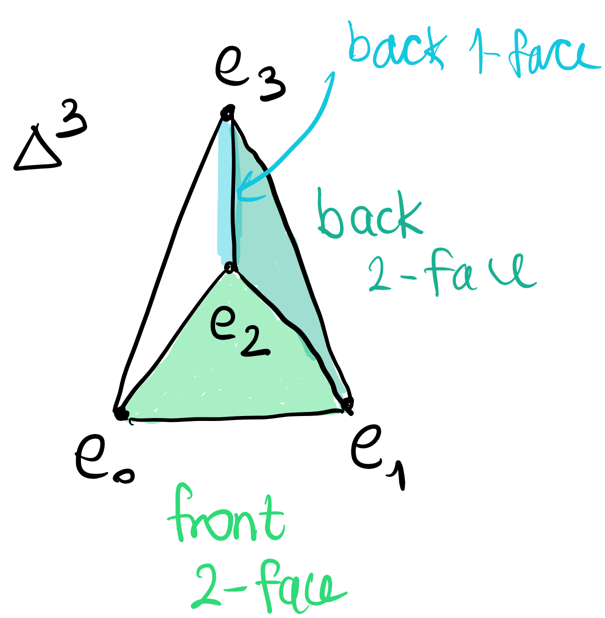
$$[e_0, e_1, \dots, e_k] : \Delta^k \to \Delta^n$$
$$e_i \longmapsto e_i \quad \text{for } i = 0, \dots, k.$$

The back k-face of Δ^n is the singular k-simplex

$$[e_{n-k}, \dots, e_n] : \Delta^k \to \Delta^n$$
$$e_i \longmapsto e_{n-k+i} \quad \text{for } i = 0, \dots, k.$$

EXAMPLE





Note: the front k-face and the back $(n-k)$-face share a single vertex:

$$[e_0,\dots,e_k] \ \& \ [e_k,\dots,e_n]. \qquad \sigma\in S_n(X)$$

For a singular simplex $\sigma:\Delta^n\to X$ its **front/ back k-face** is

$\sigma\circ[e_0,\dots,e_k]$ / $\sigma\circ[e_{n-k},\dots,e_n]$

DEFINITION

Given $\alpha\in S^k(X)$, $\beta\in S^l(X)$ and $\sigma\in S_{k+l}(X)$ let

$$(\alpha\cup\beta)(\sigma):=\alpha(\sigma\circ[e_0,\dots,e_k])\cdot\beta(\sigma\circ[e_k,\dots e_{k+l}])$$

then extend by linearity to get $\alpha\cup\beta\in S^{k+l}(X)$.

$\cup$ maps cocycles to cocycles and induces a product on cohomology.

LEMMA

For $\alpha\in S^k(X)$, $\beta\in S^l(X)$ we have

$$\delta(\alpha\cup\beta)=(\delta\alpha)\cup\beta+(-1)^k\alpha\cup(\delta\beta).$$

PROOF

Let $\sigma \in S_{k+l+1}(X)$.

- $\delta(\alpha \cup \beta)(\sigma) = (\alpha \cup \beta)(\partial \sigma) =$

$$= \sum_{i=0}^{k+l+1} (-1)^i (\alpha \cup \beta)(\sigma \circ [e_0, \dots, \hat{e}_i, \dots, e_{k+l+1}])$$

$$= \sum_{i=0}^{k} (-1)^i \alpha(\sigma \circ [e_0, \dots, \hat{e}_i, \dots, e_{k+1}])\, \beta(\sigma \circ [e_{k+1}, \dots, e_{k+l+1}]) + \sum_{i=k+1}^{k+l+1} (-1)^i \alpha(\sigma \circ [e_0, \dots, e_k])\, \beta(\sigma \circ [e_k, \dots, \hat{e}_i, \dots, e_{k+l+1}])$$

- $(\delta\alpha) \cup \beta(\sigma) = \underbrace{(\delta\alpha)(\sigma \circ [e_0, \dots, e_{k+1}])}_{\sum_{i=0}^{k+1} (-1)^i \alpha(\sigma \circ [e_0, \dots, \hat{e}_i, \dots, e_{k+1}])} \cdot \beta(\sigma \circ [e_{k+1}, \dots, e_{k+l+1}])$

for $i = k+1$

$$(-1)^{k+1} \alpha(\sigma \circ [e_0, \dots, e_k])\, \beta(\sigma \circ [e_{k+1}, \dots, e_{k+l+1}])$$

- $(-1)^k (\alpha \cup \delta\beta)(\sigma) = (-1)^k \alpha(\sigma[e_0, \dots, e_k]) \underbrace{(\delta\beta)(\sigma \circ [e_k, \dots, e_{k+l+1}])}_{\sum_{i=k}^{k+l+1} (-1)^{i-k} \beta(\sigma \circ [e_k, \dots, \hat{e}_i, \dots, e_{k+l+1}])}$

(the sign comes from the outside)

for $i = k$

$$(-1)^k \alpha(\sigma[e_0, \dots, e_k]) \cdot \beta(\sigma \circ [e_{k+1}, \dots, e_{k+l+1}])$$

The two extra terms in dark green cancel. The rest of the terms are exactly the same as above.

COROLLARY

(1) $\cup : Z^k \times Z^\ell \to Z^{k+\ell}$ ↙ cycles

(2) $\cup (B^k \times Z^\ell), \cup (Z^k \times B^\ell) \subseteq B^{k+\ell}$

(3) cup product on cochains induces cup product in cohomology:

$$\cup : H^k(X) \times H^\ell(X) \to H^{k+\ell}(X).$$

This product is associative and distributive.

(4) If $1 \in R$ is identity element, then $1 \in H^0(X;R)$ is also identity element for $\cup$

given by coeycle with value 1 on all 0-simplices

(5) the same formula defines also relative cup product.

Proof

(1) If $\delta\alpha = 0$ and $\delta\beta = 0 \Rightarrow \delta(\alpha \cup \beta) = 0$.

(2) If $\alpha = \delta\eta \Rightarrow \alpha \cup \beta = (\delta\eta) \cup \beta =$

$$= \delta(\eta \cup \beta) \pm \underbrace{\eta \cup \delta\beta}_{=0}$$

(3) Need to verify that cup product is well-defined on equivalence classes:

$$(\alpha + \delta m) \cup \beta \underset{\nearrow}{=} \alpha \cup \beta + \delta(\quad)$$

distributivity

Associativity & distributivity is clear on the level of cochains & it passes to cohomology.

(4) $(1 \cup \alpha)(\sigma) = 1(\sigma(e_0)) \cdot \alpha(\sigma)$

$$= \alpha(\sigma)$$

(5) $S^n(X, A)$ consists of cochains vanishing on simplices in A.

Consider $\cup : S^k(X, A) \times S^\ell(X) \to S^{k+\ell}(X, A)$

If $\sigma \in S_{k+\ell}(A)$,

$$(\alpha \cup \beta)(\sigma) = \underbrace{\alpha(\sigma \circ \overset{\in S_k(A)}{[\ldots]})}_{\overset{\shortparallel}{0}} \beta(\sigma \circ [\ldots]).$$

Since cup product is associative & distributive, it is natural to try to make it the multiplication in a ring structure on cohomology groups of a space X.

For this we define

$$H^*(X;R) = \bigoplus_{n\geq 0} H^n(X;R)$$

Elements of $H^*(X;R)$ are finite sums $\sum \alpha_i$ with $\alpha_i \in H^i(X;R)$, and the product of two such sums is defined to be

$$\left(\sum_i \alpha_i\right)\left(\sum_j \beta_j\right) = \sum_{i,j} \alpha_i \beta_j.$$

This makes $H^*(X;R)$ into a ring, with identity if R has identity (see (4) of the previous corollary).

This ring is called the COHOMOLOGY RING.

Taking scalar multiplication into account it is actually an R-algebra.
Coefficients may be in any ring R, which we usually suppres in the notation.

THEOREM

Let R be a commutative ring.
Then $(H^*(X,A;R),\cup)$ is graded commutative i.e. for $\alpha \in H^k(X,A)$, $\beta \in H^l(X,A)$: $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$.

Follows from the 'general' cup product statement.

THEOREM (NATURALITY OF CUP PRODUCT)

Let $f:(X,A)\to(Y,B)$ be a continuous map of pairs. Then the following diagram commutes:

$$\begin{tikzcd}
H^k(X,A)\times H^\ell(X) \arrow[r, "\cup"] & H^{k+\ell}(X,A) \\
H^k(Y,B)\times H^\ell(Y) \arrow[r, "\cup"] \arrow[u, "f^*\times f^*"] & H^{k+\ell}(Y,B) \arrow[u, "f^*"]
\end{tikzcd}$$

ie. $f^*(\alpha\cup\beta) = f^*\alpha \cup f^*\beta$.

Proof

$f_c : S_\bullet(X) \to S_\bullet(Y)$, $\sigma \longmapsto f\circ\sigma$

$f^c : S^\bullet(Y) \longrightarrow S^\bullet(X)$, $\alpha \longmapsto \alpha\circ f$

$f^c(\alpha\cup\beta)(\sigma) = (\alpha\cup\beta)(f\circ\sigma) =$

$= \alpha(f\circ\sigma\circ[e_0,\dots,e_k])\,\beta(f\circ\sigma\circ[e_k,\dots,e_{k+\ell}])$

$= f^c(\alpha)(\sigma\circ[e_0,\dots,e_k])\,f^c(\beta)(\sigma\circ[e_k,\dots,e_{k+\ell}])$

$= (f^c(\alpha)\cup f^c(\beta))(\sigma)$