CUP PRODUCTS: CLOSED SURFACES

CLASSIFICATION THEOREM

Any closed surface X is homeomorphic to either S^2, a connected sum of n tori nT, $n \geq 1$ if n is orientable, or a connected sum of projective planes mP, $m \geq 1$ if p is nonorientable. (X is connected)

The only interesting cup-product is

$H^1(X) \times H^1(X) \xrightarrow{\cup} H^2(X)$.

Graded commutativity: $\beta \cup \alpha = -\alpha \cup \beta$
(we also say it is skew-symmetric)

$\Rightarrow \alpha^2 = \alpha \cup \alpha = -\alpha^2$: $2\alpha^2 = 0$:

α^2 is a 2-torsion element
(same thing holds if we have elements of odd dimension)

$X \approx S^2$: $H_1(S^2; \mathbb{Z}) = 0 \Rightarrow H^1(S^2; \mathbb{Z}) = 0$

follows from UCT

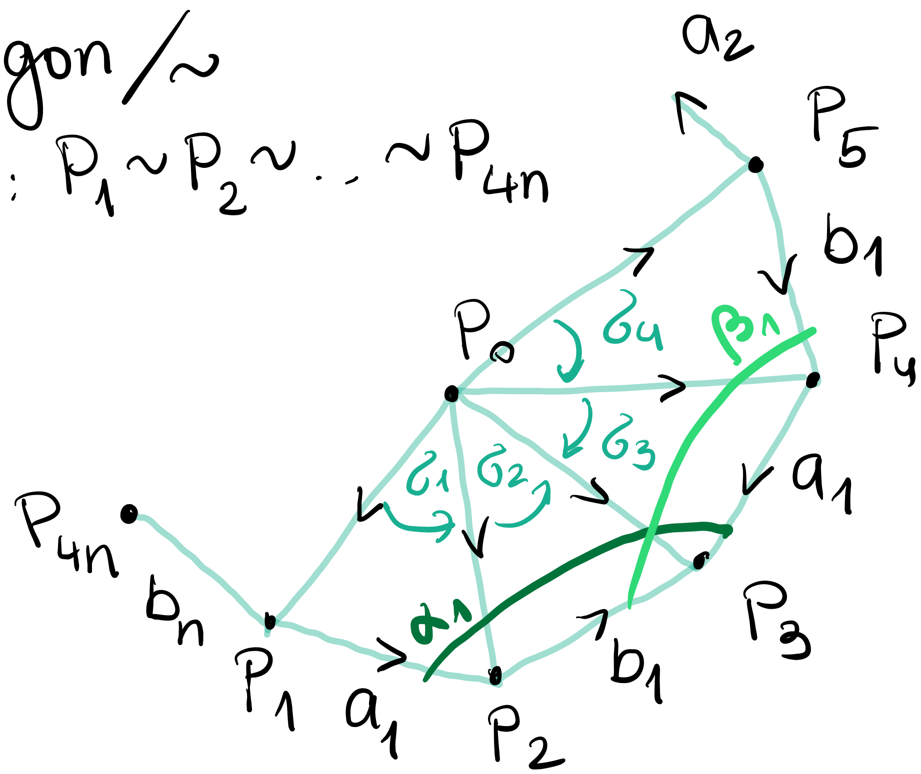
Remark: We defined $\cup$ using singular chains but clearly the same formula works for simplicial/Δ-complexes: these have the advantage of producing smaller chain groups which enable us to actually compute examples.

$X \approx nT$, $n \geq 1$

X admits a structure of Δ-cx with $4n$ 2-simplices.

$X = 4n\text{-gon}/\sim$

vertices: $P_1 \sim P_2 \sim \dots \sim P_{4n}$



order:
$P_0 < P_i \; i > 0$
$P_1 < P_2 < P_3$
$P_5 < P_4 < P_3$,

After identification all points become equivalent.

homology class of P_1 (any vertex works)

$$H_0(X) = \mathbb{Z}[P_1]$$

$$H_1(X) = \bigoplus_{i=1}^{n} \mathbb{Z}[a_i] \oplus \mathbb{Z}[b_i]$$

$$H_2(X) = \mathbb{Z}[\tau], \quad \tau = \sigma_1 + \sigma_2 - \sigma_3 - \sigma_4 + \sigma_5 + \dots - \sigma_{4n}$$

Since all of these groups are free

$$H^k(X) \cong \mathrm{Hom}(H_k(X), \mathbb{Z})$$

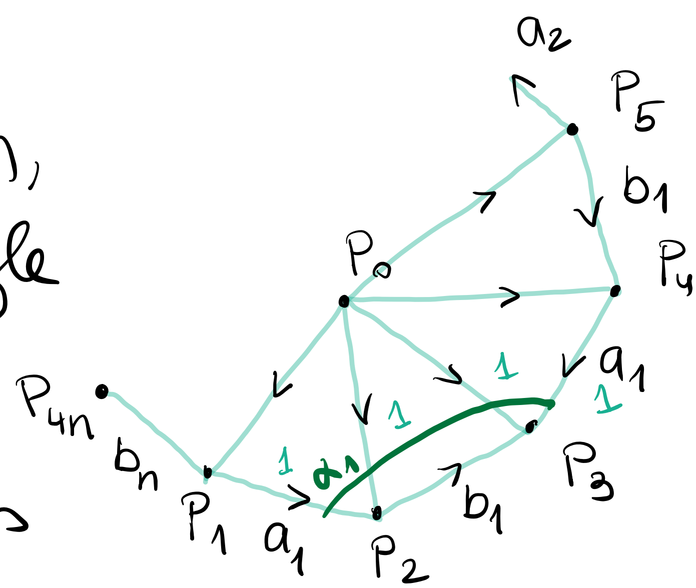
UCT
(homology groups are free)

$$H^0(X) = \mathbb{Z}\cdot 1$$ 1 function assigning 1 to any point

$$H^1(X) = \bigoplus_{i=1}^{n} \mathbb{Z}[\alpha_i] \oplus \mathbb{Z}[\beta_i]$$

$$[\alpha_i] = [a_i]^*$$
$$[\beta_i] = [b_i]^*$$

α_i has to satisfy the cocycle condition, so for each triangle $[v_0, v_1, v_2]$ the sum of its values on $[v_0, v_1]$ and $[v_1, v_2]$ equals its value on $[v_0, v_2]$. Same for β_i.



We can choose

$$\alpha_1 = \begin{cases} 1 & ; \text{ on } a_1, [P_0, P_2], [P_0, P_3] \\ 0 & ; \text{ otherwise} \end{cases}$$

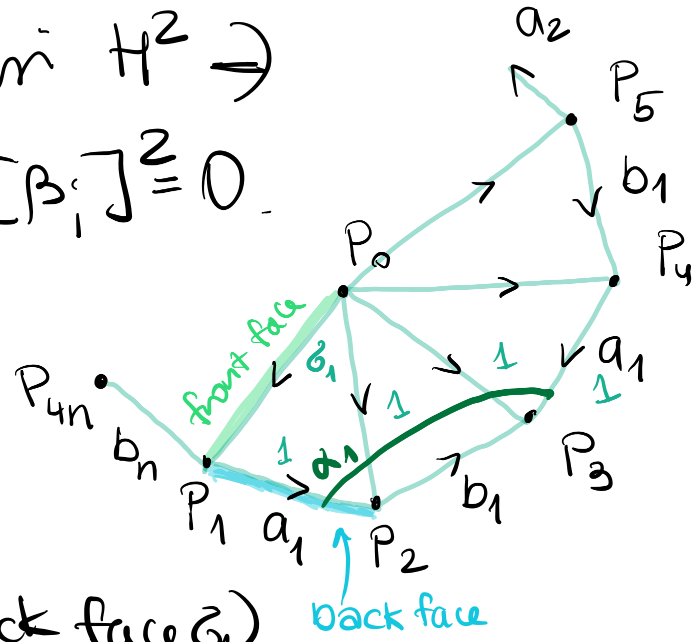
$$\beta_1 = \begin{cases} 1 & ; \text{ on } b_1, [P_0, P_3], [P_0, P_4] \\ 0 & ; \text{ otherwise} \end{cases}$$

Similar for all α_i, β_i, $i = 1, \dots, n$.

$$H^2(X) = \mathbb{Z}[m], \quad m = \tau^*.$$

Since no 2-torsion in H^2 $\Rightarrow$

$$[\alpha_i]^2 = 0, \; [\beta_i]^2 = 0.$$



$$\alpha_1 \cup \beta_1(\sigma_1) =$$
$$= \alpha_1(\text{front face } \sigma_1) \cdot \beta_1(\text{back face } \sigma_1)$$
$$= 0 \cdot 0 = 0$$

$$\alpha_1 \cup \beta_1(\sigma_2) =$$
$$= \alpha_1(\text{front face } \sigma_2) \cdot \beta_1(\text{bace face } \sigma_2) =$$
$$= 1 \cdot 1 = 1$$

We continue & get

i	1	2	3	4	>4
$(\alpha_1 \cup \beta_1)(\sigma_i)$	0	1	0	0	0
$(\beta_1 \cup \alpha_1)(\sigma_i)$	0	0	1	0	0

Why is this not skew symmetric? They're not the same on the cochain level! In cohomology they should differ by a sign.

$\delta_1+\delta_2-\delta_3-\delta_4+\delta_5+\dots$

We compute $(\alpha_1 \cup \beta_1)(\tau) = 1 \Rightarrow \alpha_1 \cup \beta_1 = \gamma = \tau^*$

? this really is the dual

$(\beta_1 \cup \alpha_1)(\tau) = -1 \Rightarrow \beta_1 \cup \alpha_1 = -\gamma$

SUMMARY

$[\alpha_i] \cup [\beta_i] = [\gamma] = -[\beta_i] \cup [\alpha_i] \quad i=1,\dots,n$

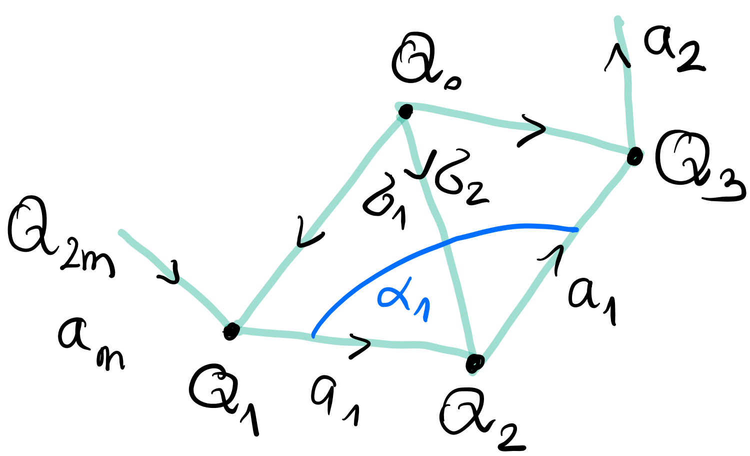
all other products vanish:

$[\alpha_i] \cup [\alpha_j] = 0, \ [\beta_i] \cup [\beta_j] = 0 \quad \forall i,j$

$[\alpha_i] \cup [\beta_j] = 0, \ [\beta_i] \cup [\alpha_j] = 0$ for $i \neq j$.

$X \approx mP, \ m \geq 1$

X admits a structure of Δ-complex with $2m$ 2-simplices, $X = 2m\text{-gon}/\sim$.



vertices: $Q_1 \sim Q_2 \sim \dots \sim Q_{2m}$

$\mathbb{Z}_2$ - coefficients

$H_0(X;\mathbb{Z}_2) = \mathbb{Z}_2[Q_1]$

$H_1(X;\mathbb{Z}_2) = \bigoplus_{i=1}^{m} \mathbb{Z}_2[a_i]$

$H_2(X;\mathbb{Z}_2) = \mathbb{Z}_2[\tau]$

$\tau = \sum_{i=1}^{2m} \sigma_i \quad \left(\partial\tau = 2\sum_{i=1}^{m} a_i = 0\right)$

Also,

$H^0(X;\mathbb{Z}_2) = \mathbb{Z}_2 \cdot 1$

$H^1(X;\mathbb{Z}_2) = \bigoplus_{i=1}^{m} \mathbb{Z}_2[\alpha_1]$

$\alpha_1 = \begin{cases} 1 & ; \text{ on } a_1, [Q_0, Q_2] \\ 0 & ; \text{ otherwise} \end{cases}$

$H^2(X;\mathbb{Z}_2) = \mathbb{Z}_2[\mu] \text{ , } \mu = \tau^*$

$(\alpha_1 \cup \alpha_1)(b_1) = 0 \qquad (\alpha_1 \cup \alpha_1)(t) = 1$

$(\alpha_1 \cup \alpha_1)(b_2) = 1 \qquad \Rightarrow \alpha_1 \cup \alpha_1 = m$

SUMMARY

$[\alpha_i] \cup [\alpha_i] = [\alpha_i]^2 = [m] \qquad i = 1, \dots, m$

$[\alpha_i] \cup [\alpha_j] = 0 \quad \text{for } i \neq j$

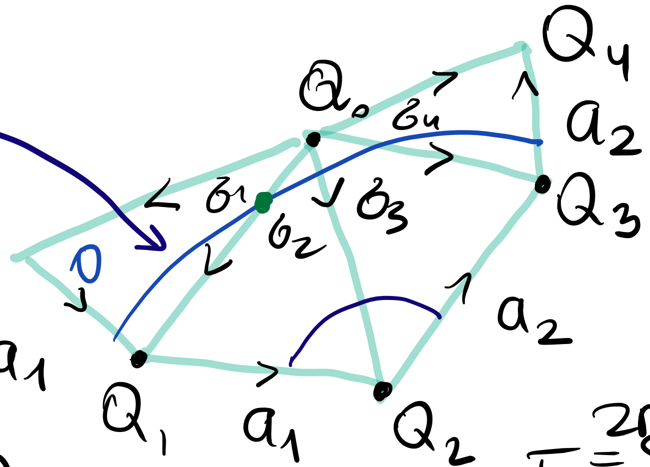
$\mathbb{Z}$ - coefficients

$$H^i(X; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2[m] & i = 2 \\ \bigoplus_{i=1}^{m-1} \mathbb{Z}[\beta_i] & i = 1 \\ \mathbb{Z} \cdot 1 & i = 0 \end{cases}$$

We can compute cup product on $H^1(X; \mathbb{Z})$ either directly (geometrically) as before or via comparison with $\mathbb{Z}_2$ - cup.

- geometric

β_i



$i = 1, \dots, m-1$

$$\tau = \sum_{i=1}^{2m} \sigma_i$$

$$\beta_1 = \begin{cases} 1 & a_1, Q_0Q_1, Q_0Q_3 \\ 2 & Q_0Q_2 \\ -1 & a_2 \end{cases}$$

$$\left.\begin{aligned} &(\beta_1 \cup \beta_1)([Q_0, Q_3, Q_4]) = -1 \\ &(\beta_1 \cup \beta_1)([Q_0, Q_1, Q_2]) = \\ &= \beta_1([Q_0, Q_1]) \cdot \beta_1([Q_1, Q_2]) \\ &= 1 \cdot 1 = 1 \\ &(\beta_1 \cup \beta_1)([Q_0, Q_2, Q_3]) = -2 \end{aligned}\right\} \quad (\beta_1 \cup \beta_1)(\tau) = -2$$

$$\Downarrow$$

$$\beta_1 \cup \beta_1 = -2m$$

- algebraic (comparison with $\mathbb{Z}_2$-coeff)

Coefficient homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_2$, $n \mapsto n \pmod 2$ induces a cochain

map $\varphi_c : S^\bullet(X; \mathbb{Z}) \to S^\bullet(X; \mathbb{Z}_2)$ and

is in turn a homomorphism on

cohomology $\varphi_*: H^*(X;\mathbb{Z}) \longrightarrow H^*(X;\mathbb{Z}_2)$.

$$\begin{tikzcd}
H^1(X;\mathbb{Z}) \times H^1(X;\mathbb{Z}) \arrow[r, "\cup"] \arrow[d, "\varphi_* \times \varphi_*"] & H^2(X;\mathbb{Z}) = \mathbb{Z}_2[\gamma] \arrow[d, "\varphi_* = id"] \\
H^1(X;\mathbb{Z}_2) \times H^1(X;\mathbb{Z}_2) \arrow[r, "\cup"'] & H^2(X;\mathbb{Z}_2) = \mathbb{Z}_2[\gamma]
\end{tikzcd}$$

What this tells us:

Since $[\beta_i]$ evaluates to 1 on $[a_i]$ and to -1 on $[a_{i+1}]$,

$$\varphi_*([\beta_i]) = [\alpha_i] - [\alpha_{i+1}] = [\alpha_i] + [\alpha_{i+1}]$$

From the commutativity of the above diagram we get

$$\begin{tikzcd}
([\beta_i], [\beta_j]) \arrow[r, mapsto, "\cup"] \arrow[d, mapsto, "\varphi_* \times \varphi_*"] & {[\beta_1] \cup [\beta_j]} \arrow[d, mapsto, "\varphi_*"] \\
([\alpha_i] + [\alpha_{i+1}], [\alpha_j] + [\alpha_{j+1}]) \arrow[r, mapsto] & {[\alpha_i] \cup [\alpha_j] + [\alpha_i] \cup [\alpha_{j+1}] + [\alpha_{i+1}] \cup [\alpha_j] + [\alpha_{i+1}] \cup [\alpha_{j+1}]}
\end{tikzcd}$$

We may assume $i \leq j$ since changing the order changes the sign which can't be seen mod 2.

$i=j$: $[\alpha_i]^2 + [\alpha_{i+1}]^2 = 2[m] = 0$

$j=i+1$: $[\alpha_{i+1}] \cup [\alpha_j] = [m]$

$j>i+1$: 0

SUMMARY

$[\beta_i] \cup [\beta_{i+1}] = [m] = [\beta_{i+1}] \cup [\beta_i]$

$i = 1, \dots, m-2$

$[\beta_i] \cup [\beta_j] = 0$, for $|i-j| \neq 1$.

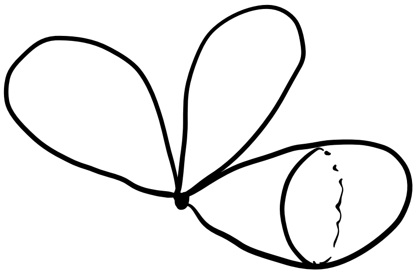
EXAMPLE

Product structure can distinguish homotopy type when groups fail to do so.

$X = S^1 \times S^1 = T$

$$
\begin{array}{lcccl}
 & 0 & 1 & 2 & \\
H^*(X): & \mathbb{Z}, & \mathbb{Z}^2, & \mathbb{Z} & \text{nontrivial} \\
 & & \mathbb{Z}\alpha \oplus \mathbb{Z}\beta & \mathbb{Z}m & \text{product} \quad \alpha \cup \beta = m
\end{array}
$$

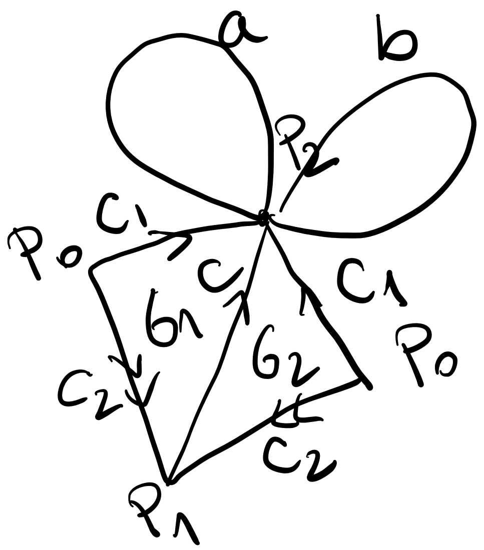
$Y = S^1 \vee S^1 \vee S^2$



$$\begin{array}{cccc} & 0 & 1 & 2 \\ H^*(Y): & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} \end{array}$$

product is trivial

Δ-complex structure



Simplicial chain complex

$$0 \to \mathbb{Z}[\sigma_1, \sigma_2] \xrightarrow{\partial_2} \mathbb{Z}[c, c_1, c_2, a, b] \xrightarrow{\partial_1} \mathbb{Z}[p_0, p_1, p_2] \to 0$$

$$\begin{array}{c|c} & \begin{array}{cc} \sigma_1 & \sigma_2 \end{array} \\ \begin{array}{c} c \\ c_1 \\ c_2 \\ a \\ b \end{array} & \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} & \begin{array}{ccccc} c & c_1 & c_2 & a & b \end{array} \\ \begin{array}{c} p_0 \\ p_1 \\ p_2 \end{array} & \begin{bmatrix} 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

dualize

$\tau = \sigma_1 + \sigma_2$

$$0 \leftarrow \mathbb{Z}[\sigma_1^*, \sigma_2^*] \xleftarrow{\delta_2} \mathbb{Z}[c^*, c_1^*, c_2^*, a^*, b^*] \xleftarrow{\delta_1} \mathbb{Z}[p_0^*, p_1^*, p_2^*] \leftarrow 0$$

$$
\delta_1 = \begin{array}{c} \\ c^* \\ c_1^* \\ c_2^* \\ a^* \\ b^* \end{array}
\begin{array}{c} \begin{array}{ccc} p_0^* & p_1^* & p_2^* \end{array} \\ \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}
\sim
\begin{array}{c} \\ c^* \\ c_1^* \\ c_2^* \\ a^* \\ b^* \end{array}
\begin{array}{c} \begin{array}{ccc} p_0^*+p_1^*+p_2^* & p_1^* & p_2^* \end{array} \\ \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}
$$

$$H^0(Y;\mathbb{Z}) \cong \mathbb{Z}\cdot 1$$

$$\operatorname{Im}\delta_1 = \langle c^* + c_1^*, c_2^* - c^* \rangle$$

$$
\delta_2 = \begin{array}{c} \\ \sigma_1^* \\ \sigma_2^* \end{array}
\begin{array}{c} \begin{array}{ccccc} c^* & c_1^* & c_2^* & a^* & b^* \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \end{bmatrix} \end{array}
\sim
\begin{array}{c} \begin{array}{ccccc} c^* & c_1^*+c^* & c_2^*-c^* & a^* & b^* \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}
$$

$$\ker\delta_2 = \langle c_1^* + c^*, c_2^* - c^*, a^*, b^* \rangle$$

$$H^1(Y;\mathbb{Z}) \cong \ker\delta_2 / \operatorname{Im}\delta_1 = \langle c_1^*+c^*, c_2^*-c^*, a^*, b^* \rangle / \langle c_1^*+c^*, c_2^*-c^* \rangle$$

$$\cong \mathbb{Z}[a^*, b^*] = \mathbb{Z}[\alpha, \beta]$$

$$\alpha = a^*$$
$$\beta = b^*$$

$$H^2(Y;\mathbb{Z}) \cong \mathbb{Z}[\tau^*] = \mathbb{Z}[\gamma]$$

$$\alpha \cup \beta(\sigma_1) = \alpha(\sigma_1[p_0,p_1])\,\beta(\sigma_1[p_1,p_2]) =$$
$$= 0$$

$$\left.\begin{array}{l} \alpha\cup\beta(\sigma_1) = 0 \\ \alpha \cup \beta(\sigma_2) = 0 \end{array}\right\} \alpha\cup\beta = 0$$

Also, $\alpha \cup \alpha = 0$ & $\beta \cup \beta = 0$.