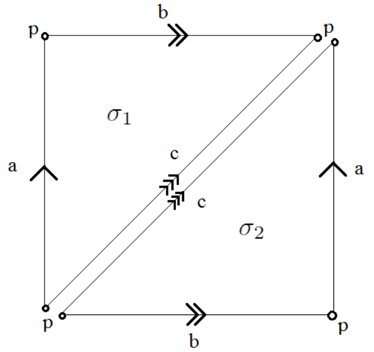
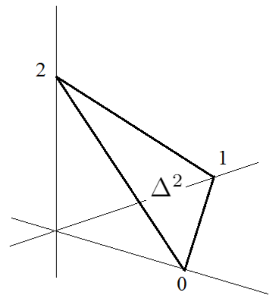
Exercise: Using the Δ-complex structure of torus T calculate its cohomology ring.

Solution We use the following Δ-structure:





The associated homology Δ-complex is

$$0 \to \mathbb{Z}(\sigma_1, \sigma_2) \stackrel{d_2}{\to} \mathbb{Z}(a, b, c) \stackrel{d_1}{\to} \mathbb{Z}(p) \to 0$$

with

$$d_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad d_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$$

and we have already seen that

$$H_0(T) = \mathbb{Z}(p) \quad H_1(T) = \mathbb{Z}(a) \oplus \mathbb{Z}(b) \quad H_2(T) = \mathbb{Z}(\sigma_1 - \sigma_2)\,.$$

Turning to cohomology we get (using the dual basis)

$$0 \leftarrow \mathbb{Z}(\bar{\sigma}_1, \bar{\sigma}_2) \stackrel{\delta_2}{\leftarrow} \mathbb{Z}(\bar{a}, \bar{b}, \bar{c}) \stackrel{\delta_1}{\leftarrow} \mathbb{Z}(\bar{p}) \leftarrow 0$$

with

$$\delta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \delta_2 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}.$$

So

$$H^0(T) = \mathbb{Z}(\bar{p})$$
$$H^1(T) = \ker \delta_2 = \mathbb{Z}(\bar{a} + \bar{c}) \oplus \mathbb{Z}(\bar{b} + \bar{c})\,.$$

Note that

$$(\bar{a} + \bar{c})(a) = 1 \quad (\bar{a} + \bar{c})(b) = 0$$

and

$$(\bar{b} + \bar{c})(a) = 0 \quad (\bar{b} + \bar{c})(b) = 1\,,$$

so the cocycle $\bar{a} + \bar{c}$ corresponds to $\bar{a} = h(\bar{a} + \bar{c}) \in \mathrm{Hom}(H_1(T) \cong \mathbb{Z}(a) \oplus \mathbb{Z}(b), \mathbb{Z})$. Similarly $\bar{b} + \bar{c}$ corresponds to homomorphism $\bar{b} \in \mathrm{Hom}(H_1(T), \mathbb{Z})$. So

$$H^1(T; \mathbb{Z}) \cong \mathbb{Z}(\bar{a}) \oplus \mathbb{Z}(\bar{b})\,.$$

(Here $\bar{a} : H_1(T) \to \mathbb{Z}$ and $\bar{b} : H_1(T) \to \mathbb{Z}$)

Next

$$H^2(T; \mathbb{Z}) \cong \mathbb{Z}^2 / \mathbb{Z}(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) \cong \mathbb{Z}(\bar{\sigma}_1)\,.$$

(in H^2 $\bar{\sigma}_1 = -\bar{\sigma}_2$) Let us see how the cup product works. The 0-cocycle $\bar{p}$ is the unit:

$$(\bar{p} \cup (\bar{a}+\bar{c}))(a) = \bar{p}(p)(\bar{a}+\bar{c})(a) = 1$$
$$(\bar{p} \cup (\bar{a}+\bar{c}))(b) = \bar{p}(p)(\bar{a}+\bar{c})(b) = 0$$
$$(\bar{p} \cup (\bar{a}+\bar{c}))(c) = \bar{p}(p)(\bar{a}+\bar{c})(c) = 1$$

so $\bar{p} \cup (\bar{a}+\bar{c}) = \bar{a}+\bar{c}$. Similarly $\bar{p} \cup (\bar{b}+\bar{c}) = \bar{b}+\bar{c}$.

$$\bar{p} \cup (\bar{\sigma}_1)(\sigma_1) = \bar{p}(p)\bar{\sigma}_1(\sigma_1) = 1$$
$$\bar{p} \cup (\bar{\sigma}_1)(\sigma_2) = \bar{p}(p)\bar{\sigma}_1(\sigma_2) = 0$$

so $\bar{p} \cup (\bar{\sigma}_1) = \bar{\sigma}_1$. Next

$$(\bar{a}+\bar{c}) \cup (\bar{a}+\bar{c})(\sigma_1) = (\bar{a}(a)+\bar{c}(a))(\bar{a}(b)+\bar{c}(b)) = 0$$
$$(\bar{a}+\bar{c}) \cup (\bar{a}+\bar{c})(\sigma_2) = (\bar{a}(b)+\bar{c}(b))(\bar{a}(a)+\bar{c}(a)) = 0$$

so $(\bar{a}+\bar{c}) \cup (\bar{a}+\bar{c}) = 0$.

$$(\bar{b}+\bar{c}) \cup (\bar{b}+\bar{c})(\sigma_1) = (\bar{b}(a)+\bar{c}(a))(\bar{b}(b)+\bar{c}(b)) = 0$$
$$(\bar{b}+\bar{c}) \cup (\bar{b}+\bar{c})(\sigma_2) = (\bar{b}(b)+\bar{c}(b))(\bar{b}(a)+\bar{c}(a)) = 0$$

so $(\bar{b}+\bar{c}) \cup (\bar{b}+\bar{c}) = 0$.

$$(\bar{a}+\bar{c}) \cup (\bar{b}+\bar{c})(\sigma_1) = (\bar{a}(a)+\bar{c}(a))(\bar{b}(b)+\bar{c}(b)) = 1$$
$$(\bar{a}+\bar{c}) \cup (\bar{b}+\bar{c})(\sigma_2) = (\bar{a}(b)+\bar{c}(b))(\bar{b}(a)+\bar{c}(a)) = 0$$

so $(\bar{a}+\bar{c}) \cup (\bar{b}+\bar{c}) = \bar{\sigma}_1$.

$$(\bar{b}+\bar{c}) \cup (\bar{a}+\bar{c})(\sigma_1) = (\bar{b}(a)+\bar{c}(a))(\bar{a}(b)+\bar{c}(b)) = 0$$
$$(\bar{b}+\bar{c}) \cup (\bar{b}+\bar{c})(\sigma_2) = (\bar{b}(b)+\bar{c}(b))(\bar{b}(a)+\bar{c}(a)) = 1$$

so $(\bar{b}+\bar{c}) \cup (\bar{b}+\bar{c}) = \bar{\sigma}_1$.

We already mentioned that $\bar{\sigma}_1 = -\bar{\sigma}_2$ which is in accordance with $(\bar{a}+\bar{c}) \cup (\bar{b}+\bar{c}) = -(\bar{b}+\bar{c}) \cup (\bar{a}+\bar{c})$.

Also $\bar{\sigma}_1(\sigma_1 - \sigma_2) = 1$, so we can view everything in H^* as elements of $\mathrm{Hom}(H_*, \mathbb{Z})$. Then

$$H^0(T;\mathbb{Z}) = \mathbb{Z}(\bar{p})$$

($\bar{p}$ being the homomorphism that maps generator of $H_0(T)$ to 1).

$$H^1(T;\mathbb{Z}) = \mathbb{Z}(\alpha) \oplus \mathbb{Z}(\beta)$$

(α being the homomorphism that maps generator a of $H_1(T)$ to 1 and b to 0 and β mapping a to 0 and b to 1).

$$H^2(T;\mathbb{Z}) = \mathbb{Z}(A)$$

(A being the homomorphism that maps generator $\sigma_1 - \sigma_2$ of $H_2(T)$ to 1).

Aditionally $\alpha \cup \alpha = \beta \cup \beta = 0$ and $\alpha \cup \beta = A = -\beta \cup \alpha$.

The result is

$$H^*(T;\mathbb{Z}) = \mathbb{Z}[\alpha, \beta]\,,$$

i.e. a free Abelian group with basis of all possible products of α, β subject to

$$\alpha^2 = 0 \quad \beta^2 = 0 \quad \beta\alpha = -\alpha\beta\,.$$