MANIFOLDS & POINCARÉ DUALITY

DEFINITION

A topological manifold of dim n is a top. space M s.t.

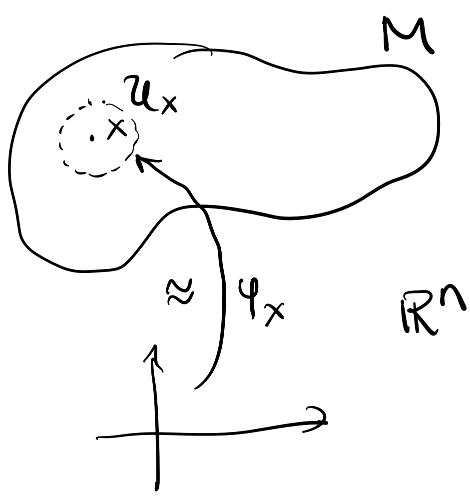
① M is Hausdorff.

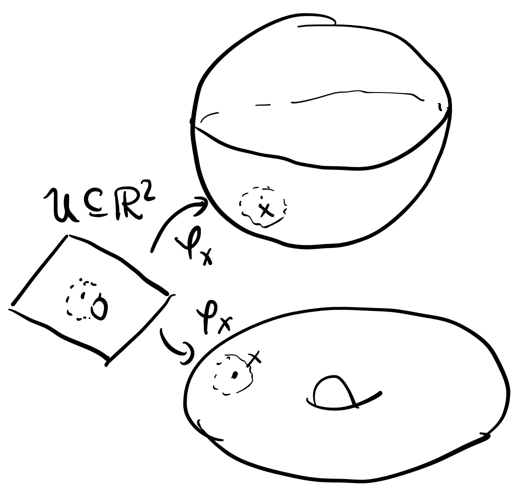
② $\forall x \in M$, $\exists$ a nbhd $U_x \subset M$ of x and a homeo $\varphi_x : \mathbb{R}^n \xrightarrow{\approx} U_x$

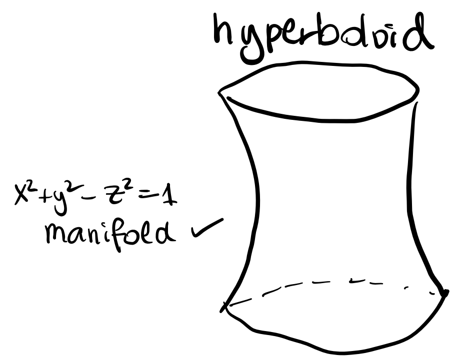
$\uparrow$ called a chart around x

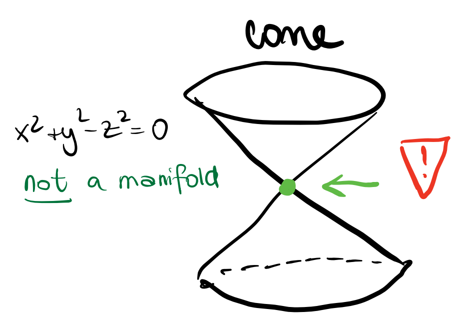
(wlog we may assume $\varphi_x(0) = x$)

(③ second countable) $\mapsto$ we won't need this









EXAMPLES

① $M = \mathbb{R}^n$, or M = open subset in $\mathbb{R}^n$ is a manifold

(2) $M = S^n$. We can cover S^n by two charts: $S^n \setminus \{N\}, S^n \setminus \{S\}$. Another option w $2n+2$ charts

$$\mathcal{U}_i^+ = \{(x_1, \dots, x_{n+1}) \in S^n : x_i > 0\}$$
$$\mathcal{U}_i^- = \{(x_1, \dots, x_{n+1}) \in S^n : x_i < 0\}$$

$i = 1, \dots, n+1$. $\mathcal{U}_i^\pm \approx \operatorname{Int} B^n(1) \approx \mathbb{R}^n$

this option is great because it gives charts on $\mathbb{R}P^n$ (take $q(\mathcal{U}_i^+) = q(\mathcal{U}_i^-)$).

(3) $\mathbb{R}P^n = S^n / \sim$ $\quad$ $q : S^n \to \mathbb{R}P^n$ $\quad$ $q|\mathcal{U}_i^\pm$ is a homeo.

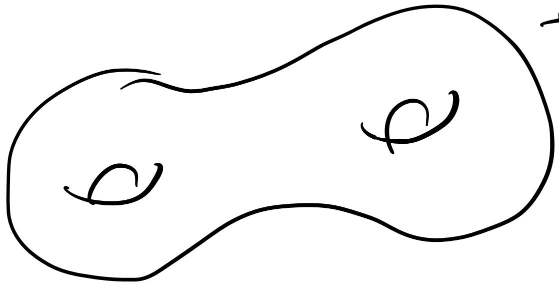
$(x \sim -x \quad \forall x \in S^n)$

$\Rightarrow \mathbb{R}P^n$ is an n-dim manifold.

(4) $M_1 = n_1$-dim manifold
$M_2 = n_2$-dim manifold

$\Rightarrow M_1 \times M_2$ is an $(n_1 + n_2)$-dim manifold.
(so $T = S^1 \times S^1 \times \dots \times S^1$ is an n-dim mnfd)

(5) A 2-dim manifold is called a surface.
This includes connected, closed surfaces



T^2 we discussed in the Examples of Cup Products section.

⑥ $M = \mathbb{C}P^n$ is a $2n$-dim manifold.

Cover $\mathbb{C}P^n$ by charts $\mathcal{U}_i$,

$$\mathcal{U}_i := \{[z_0 : \dots : z_n] : z_i \neq 0\} \subset \mathbb{C}P^n$$

$$\mathcal{U}_i \to \mathbb{C}^n \approx \mathbb{R}^{2n}$$

homogeneous coordinates

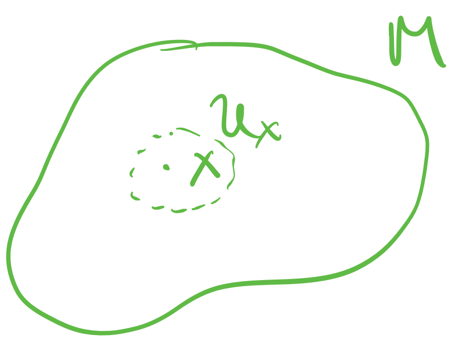
$$[z_0 : \dots : z_n] \mapsto \left(\frac{z_0}{z_i}, \dots, \frac{\hat{z}_i}{z_i}, \dots, \frac{z_n}{z_i}\right)$$

LOCAL ORIENTATION

Let M be an n-manifold. Let $x \in M$, $\varphi : \mathbb{R}^n \to \mathcal{U}_x$ a chart around x.

$$H_i(M, M \setminus \{x\}) \cong H_i(M \setminus (M \setminus \mathcal{U}_x), M \setminus \{x\} \setminus (M \setminus \mathcal{U}_x))$$

excision



using φ

$$= H_i(U_x, U_x \setminus \{x\}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

$\cong \tilde{H}_{i-1}(\mathbb{R}^n \setminus \{0\}) \cong \tilde{H}_{i-1}(S^{n-1})$

$\uparrow$ because $\tilde{H}_i(\mathbb{R}^n) = 0$

$\Rightarrow H_i(M, M\setminus\{x\}) = 0 \quad \forall i \neq n$ and

$H_n(M, M\setminus\{x\}) =$ infinite cyclic group $\cong \mathbb{Z}$

We call $H_i(M, M\setminus\{x\})$ the LOCAL HOMOLOGY of M at x.

DEFINITION

A LOCAL ORIENTATION of M at x is a choice of a generator

$$\mu_x \in \underbrace{H_n(M, M\setminus\{x\})}_{\text{infinite cyclic group}}$$

$\exists$ exactly two possible local orientations μ_x & $-\mu_x$.

REMARK (image)

If $\mathcal{U}_x \subset M$ is a chart, then μ_x induces local orientations μ_y for all $y \in \mathcal{U}_x$.

Indeed, fix $\varphi_x : \mathbb{R}^n \to \mathcal{U}_x$, let $y \in \mathcal{U}_x$ and let $B_0 \subset \mathbb{R}^n$ be a ball that contains both $\varphi_x^{-1}(x)$ & $\varphi_x^{-1}(y)$. Put $B := \varphi_x(B_0) \subset \mathcal{U}_x$.

then $H_n(M, M \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi_x^{-1}(x))$

$\cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B_0) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi_x^{-1}(y))$

$\cong H_n(M, M \setminus \{y\})$

The composition of these iso's gives us an iso

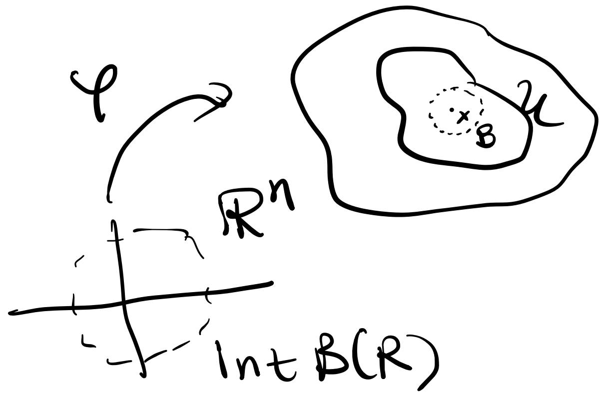
$$H_n(M, M \setminus \{x\}) \xrightarrow{\cong} H_n(M, M \setminus \{y\})$$

NOTATION: $A \subset M$ subset. We'll write $H_i(M|A; G) := H_i(M, M \setminus A; G)$ the local homology of M at A

(for $G = \mathbb{Z}$ we omit G from notation).

DEFINITION [BALL CHARTS]

$$\begin{tikzcd} x \in B \arrow[r, phantom, "\subset"] & \mathcal{U} \subset M \\ \operatorname{Int} B(R) \arrow[u, "\varphi|_{\operatorname{Int}B(r)}", "\approx"'] \arrow[r, phantom, "\subset"] & \mathbb{R}^n \arrow[u, "\varphi"', "\approx"] \end{tikzcd}$$



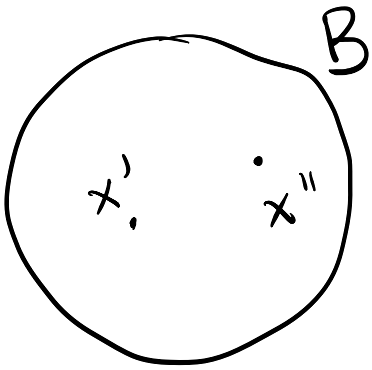
Let $B \subset M$ be a ball chart

$$H_n(M|B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B(r)) \cong \mathbb{Z}$$

depends on φ

$\forall y \in B$ we have

$$\begin{tikzcd} H_n(M, M\setminus B) \arrow[r, "inc_*", "\cong"'] \arrow[d, equal] & H_n(M, M\setminus\{y\}) \arrow[d, equal] \\ H_n(M|B) \arrow[r, "=", "L_y"'] & H_n(M|y) \end{tikzcd}$$



$\forall\, x', x'' \in B$ we get

$$\begin{tikzcd}
H_n(M|x') \arrow[rr, "\cong"] & & H_n(M|x'') \\
L_{x'} \cong & H_n(M|B) \arrow[ul] \arrow[ur] & \cong L_{x''}
\end{tikzcd}$$

ORIENTATION

Let M be an n-manifold. An ORIENTATION of M is a function $M \ni x \longmapsto \mu_x$ with $\mu_x \in H_n(M|x)$, that assigns $\forall x \in M$ a local orientation μ_x s.t. $\forall x \in M$ $\exists$ a chart $\mathcal{U}$ around x and a ball chart $B \subset \mathcal{U}$ s.t.

$$\begin{tikzcd}[column sep=small] & H_n(M|B) \arrow[dl, "L_x \cong"'] \arrow[dr, "\cong L_y"] & \\ H_n(M|x) & & H_n(M|y) \end{tikzcd}$$

$L_y L_x^{-1}(\mu_x) = \mu_y \quad \forall y \in B.$

Or, in other words, $\exists\, \mu_B \in H_n(M|B)$ a generator s.t. $L_y(\mu_B) = \mu_y \;\forall y \in B$.

If an orientation on M exists we say M is orientable. When we fix an orientation, we say M is oriented.

ORIENTATION 2-SHEETED COVER

(we do not require that a covering space $X \to Y$ is connected).

Let M be an n-manifold.

$\widetilde{M} := \{(x, \mu_x) : x \in M, \mu_x$ is a local orientation of M at x, i.e. $\mu_x \in H_n(M|x)$ is a generator$\}$

$p : \widetilde{M} \to M \quad p(x, \mu_x) := x$

2:1 map $\quad p^{-1}(x) = \{(x, \mu_x), (x, -\mu_x)\}$

Topology on $\widetilde{M}$ Let $B \subset \mathcal{U} \subset M$ be a chart and a ball chart.

Let $\mu_B \in H_n(M|B)$ be a generator.

$\forall x \in B$, we have an iso

$$H_n(M|B) \xrightarrow[\cong]{L_x} H_n(M|x).$$

Put $W(\mu_B) := \{(x, \mu_x) : x \in B, \mu_x = L_x(\mu_B)\}$

The sets $\{W(\mu_B)\}_{\mathcal{U}, \mu_B}$ form a basis of a topology on $\widetilde{M}$ (exercise).

Moreover, $p : \tilde{M} \to M$ sends $W(\mu_B)$ homeomorphically onto B.

Conclusion: $\tilde{M}$ is an n-manifold & p is a 2:1 covering.

Moreover, $\tilde{M}$ is orientable.

Indeed, an orientation on $\tilde{M}$ is given by

$$
\begin{aligned}
(x, \mu_x) \longmapsto \tilde{\mu} \in\ & H_n(\tilde{M} \mid (x, \mu_x)) \\
& \quad \cong \\
& H_n(W(\mu_B) \mid (x, \mu_x)) \\
& \quad \cong \\
& H_n(B \mid x) \cong H_n(M \mid x) \\
& \qquad\qquad\qquad \ni \mu_x
\end{aligned}
$$

where $\tilde{\mu}$ corresponds to μ_x under the above iso.

THEOREM

Assume M is a connected n-manifold. Then $\tilde{M}$ has at most two connected components. Moreover, M is orientable iff $\tilde{M}$ has two connected components.

In particular, if M is simply connected or more generally, if $\pi_1(M)$ has no subgroup of index 2, then M is orientable.

For the proof, we need the following

LEMMA

Let $p: X \to Y$ be a 2:1 covering, with Y path-connected. Then

1. X is path connected iff $\exists$ a loop in Y that lifts to a non-closed path in X.

② X can have at most two path-conn. components. When it has two, ie. $X = X' \sqcup X''$, then $p|_{X'} : X' \to Y$, $p|_{X''} : X'' \to Y$ are homeomorphisms.

We'll first prove the theorem.

Proof of theorem

Assume M is orientable. $\Rightarrow \exists$ an embedding $j : M \hookrightarrow M'$ coming from a choice of orientation

$$j(x) = (x, \mu_x)$$

and $j'(x) := (x, -\mu_x)$, $j' : M \hookrightarrow \tilde{M}$.

Clearly, j' is also an embedding.

Also, $\operatorname{im} j \cap \operatorname{im} j' = \phi \Rightarrow$

$\tilde{M} = j(M) \amalg j'(M)$.

Conversely, suppose $\tilde{M}$ is disconnected, $\tilde{M} = C_1 \amalg C_2$. By the lemma

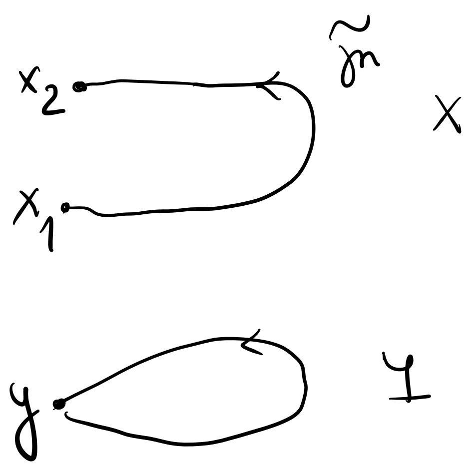
$p|_{C_1} : C_1 \to M$ is a homeo. and we obtain an orientation on M.

Now if $\pi_1(M)$ has no subgroups of index $2 \Rightarrow$ any covering $2:1$ $X \to M$ is disconnected (bc path connected coverings $d:1$ are in 1-1 correspondence with subgroups of index d of $\pi_1(M)$). ∎

Proof of Lemma

(1) Let $p: X \to Y$ be a $2:1$ covering. Take $y \in Y$. $p^{-1}(y) = \{x_1, x_2\}$ since $p: X \to Y$ is a 2-sheeted cover. We have a path $\tilde{m}$ from x_1 to x_2.

$p \circ \tilde{m}$ is a closed loop, while its lift is a non-closed path.



Conversely, suppose $\mu : I \to X$ is a loop with $\mu(0) = \mu(1) = x_0 \in Y$, and $\tilde{\mu}$ is a lift of μ with $\tilde{\mu}(0) = x_0'$, $\tilde{\mu}(1) = x_0''$, $x_0' \neq x_0''$. Now take any point $\tilde{x} \in X$. Put $x := p(\tilde{x})$. Y is path-connected, so take a path α in Y with $\alpha(0) = x$, $\alpha(1) = x_0$.

By lifting α starting at $\tilde{x}$ we get a path from $\tilde{x}$ to one of x_0' or x_0''. But x_0' & x_0'' are in the same path-connected component of X.

$\Rightarrow \tilde{x}$ is also in that component.

This proves statement 1.

(2) Suppose X is not path-connected. Let X' be a path-connected comp.

of X. Obviously, $\forall$ 2 points $x_1, x_2 \in X'$ with $x_1 \neq x_2$ we have $p(x_1) \neq p(x_2)$ otherwise we'll have a non-closed path in X' which projects under p to a loop in Y. Contradiction, by ①. Also, $p(X') = Y$ because given $y \in Y$ just choose $x_0 \in X$, put $x_0 := p(x_0')$, take a path $\gamma : I \to Y$ with $\gamma(0) = x_0$ & $\gamma(1) = y$ and now lift γ to a path $\tilde{\gamma} : I \to X$ with $\gamma(0) = x_0'$. Then $\tilde{\gamma}(1) \in X'$ & $p(\tilde{\gamma}(1)) = y$. So, $p : X' \to Y$ is 1-1. By the definition of a covering space, p is a local homeo $\Rightarrow$ p is a homeo. The fact that $\# \pi_0(X) = 2$ is straightforward. ∎

ANOTHER USEFUL COVERING SPACE

Define $\tilde{M}_{\mathbb{Z}} = \{(x, \alpha_x) : x \in M, \alpha_x \in H_n(M|x)\}$

α_x is not necessarily a generator.

$p : \tilde{M}_{\mathbb{Z}} \to M \quad p(x, \alpha_x) := x.$

Topology on $\tilde{M}_{\mathbb{Z}}$

Let $B \subset M$ be a ball chart

$W(\alpha_B) = \{(x, \alpha_x) \in \tilde{M}_{\mathbb{Z}} : x \in B, L_x(\alpha_B) = \alpha_x\}$

this is a basis for a topology on $\tilde{M}$. Inside $\tilde{M}_{\mathbb{Z}}$ we have $M_0 \cong M$,

$$M_0 = \{(x, 0) : x \in M\}$$

the rest of $M_{\mathbb{Z}}$ consists of an infinite sequence of copies M_k, where

$M_k := \{(x, \alpha_x) : x \in M, \alpha_x$ is k-times a generator of $H_n(M|x)\}$

$1 \leq k \in \mathbb{Z}$

DEFINITION

Let $X \xrightarrow{p} Y$ be a covering. A SECTION $s: Y \to X$ is a continuous map $s: Y \to X$ s.t. $p \circ s = id_X$.

$$\begin{tikzcd} X \arrow[d, "p"'] \\ Y \arrow[u, bend right, "s"'] \end{tikzcd}$$

So, an orientation on M is a section $\mu: M \to \tilde{M}$. Or, a section

$$\alpha: M \to \tilde{M}_{\mathbb{Z}} \quad \text{with } \alpha_x \in H_n(M|x)$$
$$x \mapsto \alpha_x$$

a generator $\forall x$.

A further generalization

Let R be a commutative ring with a unity $1 \in R$.

$$H_n(M|x;R) \cong R$$ free R-module of rank 1

A local R-orientation at x is

a choice of a generator $u \in R$, unit/invertible elt.

ie. $R = R \cdot u$.

Of course, two generators $u, v \in R$ differ by an invertible element

$v = b \cdot u$, $b \in R$ invertible.

Define $\tilde{M}_R$ similarly to $\tilde{M}_{\mathbb{Z}}$.

DEFINITION

An **R-ORIENTATION** on M, is a section $\mu : M \to \tilde{M}_R$ s.t. $\forall x \in M$ μ_x is a generator of $H_n(M|x; R)$.

Exercise This definition is equivalent to the previous one for $R = \mathbb{Z}$.

REMARK

$H_n(M|x; R) \cong H_n(M|x) \otimes R \Rightarrow$

inside $\tilde{M}_R$ we have $\tilde{M}_r \subset \tilde{M}_R$ $\forall r \in R$

$\tilde{M}_r = \{(x, \pm\mu_x \otimes r) : x \in M\}$

μ_x is the gen of $H_n(M|x)$

Note that if $2r=0$ (ie $r=-r$), then $\tilde{M}_r = M$. If $2r \neq 0 \Rightarrow \tilde{M}_r \approx \tilde{M}$.

Conclusion

1. If M is orientable then it is R-orientable for every ring R
2. Let M be a non-orientable manifold & R a ring with a unit of order 2 (ie $2=0$ in R) $\Rightarrow M$ is R-orientable. In particular, any manifold is $\mathbb{Z}_2$-orientable.