THEOREM

Let M be a compact connected n-manifold.

① If M is R-orientable then the map

$$H_n(M;R) \xrightarrow{L_x} H_n(M|x;R) \cong R$$

is an iso $\forall x \in M$.

② If M is <u>not</u> R-orientable, then $\forall x \in M$ the map

$$H_n(M;R) \xrightarrow{L_x} H_n(M|x;R) \cong R$$

is injective and its image is $\{a \in R : 2a = 0\}$.

③ $H_i(M;R) = 0 \quad \forall i > n$.

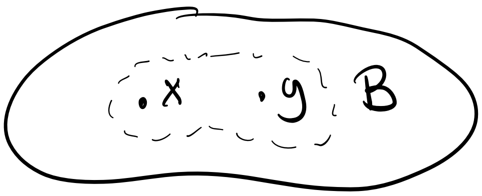
So, if M is orientable, then $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$.

If not $\Rightarrow H_n(M;\mathbb{Z}) = 0$.

REMARKS

① Suppose M is R-orientable and let μ be an R-orientation. Let $x \in M$

and consider $\mu_x \in H_n(M|x;R)$. By (1) of the Theorem we get a class $a^x \in H_n(M;R)$ s.t. $a^x \overset{L_x}{\longmapsto} \mu_x$. Consider $y \in M$, lying in the same ball chart as x.



$$\begin{tikzcd}
 & H_n(M;R) \arrow[dl, "L_x"'] \arrow[d] \arrow[dr, bend left] & \\
H_n(M|x;R) \arrow[ur, phantom, "c", shift left=6] & H_n(M|B;R) \arrow[l] \arrow[r] \arrow[ur, phantom, "c", shift left=6] & H_n(M|y;R)
\end{tikzcd}$$

Consider $\mu_y \in H_n(M|y;R)$ coming from μ. $\Rightarrow L_y(a^x) = \mu_y$.

If M is connected, all the above works even if x & y are not in the same ball chart.

Also, if $a \in H_n(M;R)$ a generator $\Rightarrow$

$M \ni x \longmapsto \mu_x := L_x(a)$ is an orientation.

So R orientations on a compact $M \longleftrightarrow$ generators of $H_n(M;R)$.

A choice of a generator, in case M is compact and orientable of $H_n(M;R)$ is called a **FUNDAMENTAL CLASS**.

NOTATION Let M be a compact R-oriented n-manifold. We denote by $[M] \in H_n(M;R)$ the fundamental class corresponding to the given orientation.

(2) If M, an n-manifold, has a class $a \in H_n(M;R)$ s.t. a induces an orientation by $x \longmapsto L_x(a)$, then M is compact.

PROOF

Let δ be a cycle representing a.

Clearly $\text{im}(\sigma) = \text{compact}$.

(union of the images of the simplices participating in σ)

So if $x \in M \setminus \text{im}(\sigma) \Rightarrow$

$$L_x([\sigma]) = 0 \in H_n(M|x;R).$$

(σ lies entirely in $M \setminus \{x\}$ and so $[\sigma]=0$ in $H_n(M|x;R)$)

$\Rightarrow \text{Im}(\sigma) = M$. ∎

To prove the theorem we need the following lemma:

LEMMA

Let M be an n-manifold. Let $A \subset M$ be a compact subset. Then

1. If $M \ni x \longmapsto \alpha_x \in H_n(M|x;R)$ is a section of $\tilde{M}_R \longrightarrow M$ then $\exists$ a unique $\alpha_A \in H_n(M|A;R)$ s.t.

$L_x(\alpha_A) = \alpha_x \quad \forall x \in A.$

② $H_i(M|A;R) = 0 \quad \forall i > n.$

Proof of the theorem (assuming the lemma)

By assumption M = compact, so we can take $A = M$ in the lemma.

$$H_K(M|A;R) = H_K(M,\phi;R) = H_K(M;R).$$

$\Rightarrow$ (3) of the theorem follows from the lemma.

Denote by Γ_R the set of sections $\tilde{M}_R \to M$. Note that Γ_R is an R-module (we can add sections and also multiply a section by $r \in R$).

exercise: these operations preserve continuity

We have a homo. $H_n(M;R) \xrightarrow{\Theta} \Gamma_R$

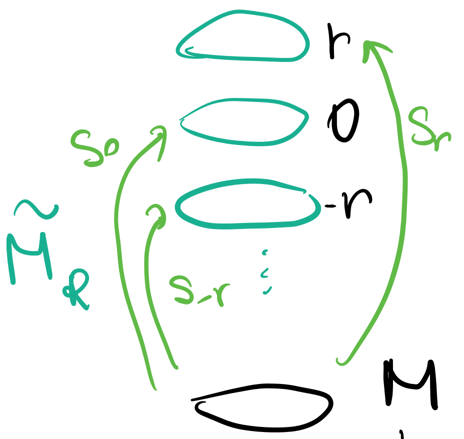
$$H_n(M;R) \ni a \overset{\Theta}{\longmapsto} \big(M \ni x \longmapsto L_x(a)\big) \in \Gamma_R.$$

By the lemma Θ is an iso.

Pick $x_0 \in M$. We have a 'restriction' map $\rho : \Gamma_R \longrightarrow H_n(M|x_0;R) = (\tilde{M}_R)_{x_0} \cong R$ fiber of $\tilde{M}_R$ over x_0

ρ is injective (follows from uniqueness of lifts in covering spaces).

If M is R-orientable then ρ is an iso. $\tilde{M}_R$ is isomorphic to R copies of



M (one copy of M for each $r \in R$). Each section s_r is a constant function & therefore determined by the value in a single point.

$$\Rightarrow \begin{tikzcd} H_n(M;R) \arrow[r, "\Theta"] \arrow[rr, bend left, "L_x"] & \Gamma_R \arrow[r, "\rho"] & H_n(M|x;R) \cong R \end{tikzcd}$$

is an iso $\forall x \in M$.

If M is <u>not</u> R-orientable, then $\Gamma_R \xrightarrow{\rho} H_n(M|x;R)$ is only injective.

Clearly, $\operatorname{im}(\rho) = \{ a \in H_n(M|x;R) : -a = a \}$,
because $\forall r \in R$ with $2r \neq 0$ we have $\tilde{M}_r \approx \tilde{M}$. If M is not R-orientable, it is not $\mathbb{Z}$-orientable.

Exercise: finish the details as an exercise.

$\blacksquare$

To prove the lemma we need the following version of M-V LES:

THEOREM

Let X be a space, $Y \subset X$ a subspace.

Let $Q, R \subset X$, $S \subset Q$, $T \subset R$, $S, T \subset Y$ s.t. $\operatorname{Int} Q \cup \operatorname{Int} R = X$, $\operatorname{Int} S \cup \operatorname{Int} T = Y$.

Then $\exists$ a LES

$$\cdots \to H_k(Q \cap R, S \cap T) \xrightarrow{\Phi} H_k(Q,S) \oplus H_k(R,T) \xrightarrow{\Psi} H_k(X,Y) \to \cdots$$

where $\Phi(x) = (x, -x)$, $\Psi(x, y) = x + y$.
The theorem works with coefficients in any group.

PROOF

See Hatcher.

COROLLARY

Let M be an n-manifold, $A, B \subset M$ compact. Then we have a LES

$$\cdots \to H_k(M|A\cup B) \xrightarrow{\Phi} H_k(M|A) \oplus H_k(M|B) \xrightarrow{\Psi} H_k(M|A\cap B) \to \cdots$$

PROOF

Take $Q = R = M = X$, $Y = M \setminus (A \cap B)$, $S = M \setminus A$, $T = M \setminus B$.

$\blacksquare$

NOTATION $B \subset A \subset X$.

$$H_k(X|A) \xrightarrow{L_{A,B}} H_k(X|B)$$

The map induced by the inclusion $(X, X\setminus A) \to (X, X\setminus B)$.

PROOF OF LEMMA

Step 1 If the lemma holds for two subsets A & B & their intersection, then it also holds for the union $A\cup B$.

Proof of Step 1 We use MV.

$$\cdots \to H_k(M|A\cup B) \xrightarrow{\Phi} H_k(M|A) \oplus H_k(M|B) \xrightarrow{\Psi} H_k(M|A\cap B) \to \cdots$$

As $H_k(M|A\cap B) = 0 \ \forall\, k \geq n+1$ by assumption, hence we get an exact sequence

$$0 \to H_n(M|A\cup B) \xrightarrow{\Phi} H_n(M|A) \oplus H_n(M|B) \xrightarrow{\Psi} H_n(M|A\cap B)$$

$\Phi(\alpha) = (\alpha, -\alpha)$ (formally, $\Phi(\alpha) = (L_{A\cup B,A}(\alpha), L_{A\cup B,B}(\alpha))$,

$\Psi(\alpha, \beta) = \alpha + \beta$ (—‖— $\Psi(\alpha, \beta) = \cdots$).

We know, by assumption, that

$H_k(M|A) = H_k(M|B) = 0 \quad \forall\, k \geq n+1$

$\Rightarrow H_k(M \mid A \cup B) = 0 \quad \forall k \geq n+1.$

This proves (2) of the lemma for $A \cup B$.

If $x \mapsto \alpha_x$ is a section of $\tilde{M}_R \to M$, then by assumption $\exists\, \alpha_A \in H_n(M|A)$, $\alpha_B \in H_n(M|B)$ s.t. $L_{A,x}(\alpha_A) = \alpha_x$, $\forall x \in A$

$L_{B,x}(\alpha_B) = \alpha_x \quad \forall x \in B.$

Consider $\alpha'_{A\cap B} := L_{A,A\cap B}(\alpha_A)$, $\alpha''_{A\cap B} := L_{B,A\cap B}(\alpha_B)$. Clearly, $L_{A\cap B,x}(\alpha'_{A\cap B}) = \alpha_x$, $L_{A\cap B,x}(\alpha''_{A\cap B}) = \alpha_x$ $\forall x \in A \cap B$.

By the uniqueness assumption we have $L_{A,A\cap B}(\alpha_A) = L_{B,A\cap B}(\alpha_B)$.

$$\begin{array}{cc} \parallel & \parallel \\ \alpha'_{A\cap B} & \alpha''_{A\cap B} \end{array}$$

Denote $\alpha_{A\cap B} := \alpha'_{A\cap B} = \alpha''_{A\cap B}$.

Clearly $\Psi(\alpha_A, -\alpha_B) = 0$. By exactness of the MV sequence

$$\exists\, \alpha_{A\cup B} \in H_n(M \mid A\cup B) \text{ s.t.}$$

$$\Phi(\alpha_{A\cup B}) = (\alpha_A, -\alpha_B) \Rightarrow$$

$$L_{A\cup B, x}(\alpha_{A\cup B}) = \alpha_x \quad \forall x \in A\cup B.$$

Uniqueness:

Enough to prove that if

$$L_{A\cup B, x}(\alpha) = 0 \quad \forall x \in A\cup B,$$

then $\alpha = 0$. Indeed, if $L_{A\cup B, x}(\alpha) = 0$ $\forall x \in A\cup B$, then $\alpha_A := L_{A\cup B, A}(\alpha)$ &

$\alpha_B := L_{A\cup B, B}(\alpha)$ also satisfy

$L_{A,x}(\alpha_A) = 0 \quad \forall x \in A$ & $L_{B,x}(\alpha_B) = 0$

$\forall x \in B$. By the uniqueness assumption we have $\alpha_A = 0, \alpha_B = 0$. But

$(\alpha_A, -\alpha_B) = \Phi(\alpha)$ & Φ is injective.

$\Rightarrow \alpha = 0$. This completes the proof of step 1.

Step 2 We'll reduce to proving the lemma to the case $M = \mathbb{R}^n$.

If $A \subset M$ is compact $\Rightarrow A = A_1 \cup .. \cup A_m$

with A_i = compact $\forall i$ & $A_i \subset$ ball chart $\subset \mathbb{R}^n$.

If the result is true for $A_1 \cup .. \cup A_{m-1}$ & also for A_m & for

union of $m-1$ compact sets each contained in $\mathbb{R}^n \subset M$

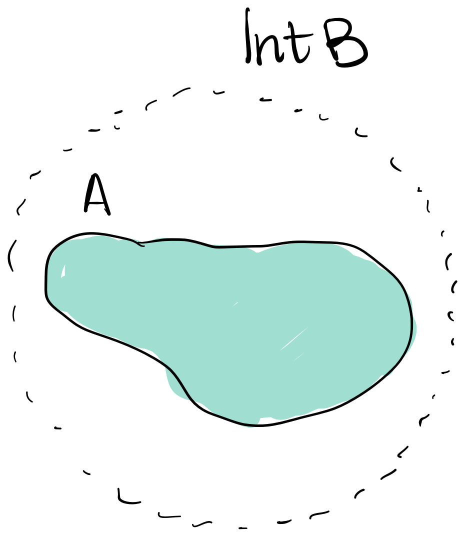
$$(A_1 \cup .. \cup A_{m-1}) \cap A_m = (A_1 \cap A_m) \cup \dots \cup (A_{m-1} \cap A_m)$$

then by step 1, the result holds also

for $A_1 \cup \dots \cup A_m$. So, by induction on m, it is enough to prove the result for $m=1$, i.e. $A \subset$ ball chart $\subset M$.

Assume that
$$A \subset \underset{\substack{\wr\wr \\ \operatorname{Int} B^n(R)}}{\operatorname{Int} B} \subset \underset{\substack{\wr\wr \\ \mathbb{R}^n}}{\mathcal{U}} \subset M$$

ball of radius R



By excision: $H_n(M, M\setminus A) \cong$

$\cong H_n(M\setminus(M\setminus \operatorname{Int} B), M\setminus A\setminus(M\setminus \operatorname{Int} B))$

$= H_n(\operatorname{Int} B, \operatorname{Int} B\setminus A) \cong H_n(\mathcal{U}|A)$

(the isos here are induced by inclusions).

Step 3

Assume $M = \mathbb{R}^n$, $A \subset M$ is compact and $A = A_1 \cup \dots \cup A_m$ with A_i convex for all i. If $m=1$ & $A=$ convex, then $H_n(M|A) \xrightarrow{\cong} H_n(M|x)$ for any $x \in A$. (the inclusion $(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus A)$ is a homotopy equivalence). If $A = A_1 \cup \dots \cup A_m$ with A_i convex, then use induction on m and the previous steps.

both strongly def. retract onto a sphere centered at x

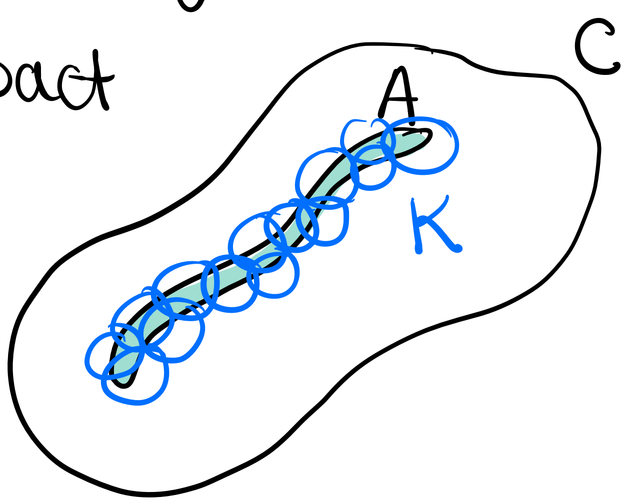
Step 4

$M = \mathbb{R}^n$, $A \subset \mathbb{R}^n$ is an arbitrary compact subset.

Let $\alpha \in H_i(M|A)$. Let z be a cycle in $S_i(M, M \setminus A)$ with $\alpha = [z]$. View z also as a chain in $S_i(M)$, and let $C :=$ union of images of all the sing Δ in ∂z

So $C \subset M \setminus A$. Clearly, C is compact.

Since A is compact and C too, $\exists\, \delta > 0$ s.t. $\forall p \in C$, $q \in A$,



$\operatorname{dist}(p,q) \geq \delta$. Now cover A by finitely many closed balls centered at points of A and with radius $< \frac{\delta}{2}$. Denote the union of these balls by K. Note that $C \subset M \setminus K \Rightarrow z$ is also a cycle in $S_i(M, M\setminus K)$. Put $\alpha_K := [z] \in H_i(M|K)$.

K is a finite union of convex sets (balls)

If $i > n$, then by Step 3, $\alpha_K = 0$. $\Rightarrow$

$\alpha = L_{K,A}(\alpha_K) = 0 \Rightarrow H_i(M|A) = 0 \quad \forall i > n$.

Now let $x \mapsto \alpha_x$ be a section $\tilde{M}_R \to M$ $(M = \mathbb{R}^n)$. Assume that $\alpha_x = L_{A,x}(\alpha)$ $\forall x \in A$ for some $\alpha \in H_n(\mathbb{R}^n | A)$. We'll show α is unique. Enough to show that if

$\alpha_x = 0$ $\forall x \Rightarrow \alpha = 0$.

CLAIM Assuming $\alpha_x = 0$ $\forall x \in A$ implies $\alpha_x = 0$ $\forall x \in K$.

Proof of this claim:

If $B \subset K$ is one of the balls forming K, then $H_n(\mathbb{R}^n | B) \xrightarrow[\cong]{L_{B,x}} H_n(\mathbb{R}^n | x)$ is an iso $\forall x \in B$.

$\Rightarrow \alpha_x = 0$ $\forall x \in A \Rightarrow \alpha_x = 0$ $\forall x \in B$

$\Rightarrow \alpha_x = 0$ $\forall x \in K$. This proves the claim. $\blacksquare$

Now define $\alpha_K \in H_n(\mathbb{R}^n | K)$ as before.

By Step 3 we get $\alpha_K = 0$. $\Rightarrow$

$\Rightarrow \alpha = L_{K,A}(\alpha_K) = 0$ too. This proves uniqueness of (1) of the lemma.

EXISTENCE

Pick a ball $B(r)$ with $\operatorname{Int} B(r) \supset A$.

By step 3, $\exists$ a class $\alpha_{B(r)} \in H_n(\mathbb{R}^n \setminus B(r))$ with $L_{B(r),x}(\alpha_{B(r)}) = \alpha_x \ \forall x \in B(r)$.

Put $\alpha_A := L_{B(r),A}(\alpha_{B(r)})$. $\blacksquare$

closed manifold = compact manifold (without boundary)

THEOREM

Let M be a connected non-compact n-manifold. Then $H_i(M;R) = 0 \ \forall i \geq n$.

Proof

See 3.29 in Hatcher.

THEOREM

Homology groups of a compact manifold are finitely generated.

Proof

See Corollaries A.8 & A.9 in Hatcher.

COROLLARY

Let M be a closed manifold, of dim n, and assume M is connected.

If M is orientable $\Rightarrow$ torsion $H_{n-1}(M) = 0$.

If M is non-orientable $\Rightarrow$ torsion $H_{n-1}(M) = \mathbb{Z}_2$

PROOF

Recall UCT:

$$0 \to H_n(M) \otimes R \to H_n(M; R) \to \operatorname{Tor}(H_{n-1}(M), R) \to 0$$

torsion $H_{n-1}(M) = \bigoplus_{i=1}^{r} \mathbb{Z}_{\ell_i}$, $\ell_i \geq 2, r \geq 0$

($r=0$ means torsion $=0$).

Assume $M =$ orientable. If torsion$(H_{n-1}(M)) \neq 0$, i.e. $r \geq 1$, choose p = prime s.t. $p \mid \ell_1$.

Take $R = \mathbb{Z}_p$.

$$0 \to \underset{\mathbb{Z}}{\underset{\shortparallel}{H_n(M) \otimes \mathbb{Z}_p}} \to \underset{\mathbb{Z}_p}{\underset{\shortparallel}{H_n(M;\mathbb{Z}_p)}} \to \bigoplus_{i=1}^{r} \underbrace{Tor(\mathbb{Z}_{\ell_i}, \mathbb{Z}_p)}_{\mathbb{Z}_{\gcd(\ell_i, p)}} \to 0$$

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_p \to \mathbb{Z}_p \oplus \dots \to 0$$

This is impossible.

Assume M is not orientable.

Take $R = \mathbb{Z}_m$.

CLAIM

$$H_n(M;\mathbb{Z}_m) \cong \{a \in \mathbb{Z}_m : 2a = 0\} = \begin{cases} 0 & m \text{ odd} \\ \underset{\mathbb{Z}_2}{\underset{\shortparallel\wr}{\{0, \frac{m}{2}\}}} & m \text{ even} \end{cases} \subset \mathbb{Z}_m.$$

PROOF

m = odd $\Rightarrow$ M is not $\mathbb{Z}_m$-orientable

because $\forall\ 0 \neq r \in \mathbb{Z}_m$, $\tilde{M}_r \approx \tilde{M}$.

m = even > 2 $\Rightarrow$ M is not $\mathbb{Z}_m$ orientable.

We get from UCT:

$$0 \to \underbrace{H_n(M)}_{0} \otimes \mathbb{Z}_m \to H_n(M;\mathbb{Z}_m) \to \bigoplus_{i=1}^{r} \mathbb{Z}_{\gcd(l_i,m)} \to 0$$

Take $m = \text{odd} \Rightarrow \gcd(l_i, m) = 1$. this holds for all $m = \text{odd} \Rightarrow l_i = 2^{s_i} \ \forall i$.

For $m = \text{even}$, $H_n(M;\mathbb{Z}_m) \cong \mathbb{Z}_2$,
hence $\bigoplus_{i=1}^{r} \mathbb{Z}_{\gcd(l_i,m)} \cong \mathbb{Z}_2$.

$\Rightarrow r = 1$ & $l_1 = 2$. $\Rightarrow$ torsion $H_{n-1}(M) = \mathbb{Z}_2$.