We'll apply the above to chain complexes of singular chains $S_\bullet(X)$, $S_\bullet(X, A)$.

We'll also apply it to the cellular chain complex $C_\bullet^{cw}(X)$, etc.

SINGULAR HOMOLOGY:

$S_\bullet(X;G) := S_\bullet(X) \otimes G$

$S_n(X;G) = \left\{ \sum_i n_i \sigma_i : \sigma_i \in \Delta^n \to X, n_i \in G \right\}$

$\partial : S_n(X;G) \to S_{n-1}(X;G)$ has the same 'formula':

$$\partial(g\sigma) = \sum_{j=0}^{n} (-1)^j \cdot g\sigma\big|_{[v_0, \dots, \hat{v_j}, \dots, v_n]}$$

$$\sigma : \Delta^n \to X, \ g \in G$$

$\partial\left(\sum n_i \sigma_i\right) := \sum_i n_i \partial^{old}(\sigma_i)$

$\partial^2 = 0$

$$H_n(X;G) := H_n(S_\bullet(X;G)).$$

$A \subset X$ subspace, $S_n(X,A;G) := \dfrac{S_n(X;G)}{S_n(A;G)}$

$\rightsquigarrow H_n(X,A;G)$.

Reduced homology $\tilde{H}_n(X;G)$ is $\quad X \neq \emptyset$
the homology of the augmented complex

$$\cdots \to S_1(X;G) \xrightarrow{\partial} S_0(X;G) \xrightarrow{\varepsilon} G \to 0$$

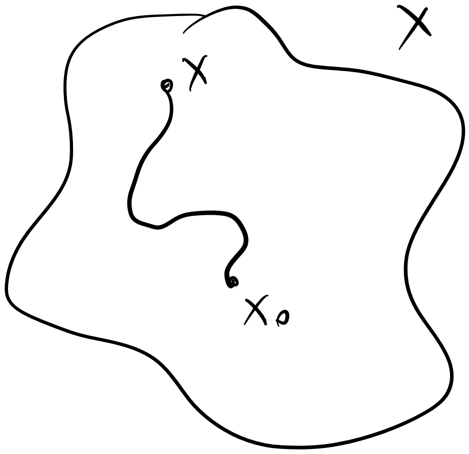
$$\sum n_i x_i \longmapsto \sum n_i$$

One can show that if X is path-connected, $H_0(X;G) \cong G$ via the following iso:

$H_0(X;G)$ is the 0-th homology of

$$\cdots \to S_1(X) \otimes G \to S_0(X) \otimes G \to 0$$

Pick $x_0 \in X$. Identify $S_0(X)$ with points:
$\sigma : \bullet \to X$ represents $x \in X$. If X is path connected, then $\exists\, \gamma$ for any x_0 & x s.t. $\gamma(0) = x_0$, $\gamma(1) = x$.



$\partial\sigma = x - x_0.$

Define $f: g \mapsto [g x_0] \in H_0(X;G).$

$(\partial \otimes id)(\sigma \otimes g) = (\partial\sigma \otimes g) = g(x - x_0) =$
$= gx - gx_0$

It follows that $[gx_0]$ is independent of our choice of x.

The generators of $S_0(X)\otimes G$ are $x \otimes g$, where x is a singular simplex & $g \in G$.

Consequently, $[gx]$ also generate $H_0(X;G)$. Since $[gx] = [gx_0]$ $\forall x$, this map is surjective.

$$\varepsilon(\partial m \otimes g) = \varepsilon(m(1) \otimes g) + \varepsilon(m(0) \otimes (-g))$$

$$= g - g = 0$$

ε induces a map on $H_0(X;G)$ $\bar{\varepsilon}$.

$$\bar{\varepsilon}(f(g)) = \bar{\varepsilon}([g x_0]) = \varepsilon(g x_0) = g.$$

$$\begin{tikzcd}
g(x_0) & \\
S_0(X) \otimes G \arrow[r, "\varepsilon"] \arrow[d, "{[g(x_0)]}"'] & G \\
H_0(X;G) \arrow[ur, "\bar{\varepsilon}"'] &
\end{tikzcd}$$

$\bar{\varepsilon}$ is the left inverse of f, so f is injective. $\Rightarrow$ f is bijective.

∎

Most of the homology theory carries over to $H_*(X,A;G)$: LES of a pair, homotopy axiom, excision, MV LES. → $H_*(X,A;G)$ is a homology theory

EXAMPLE

Let $X \neq \emptyset$ be a space, fix a basepoint $* \in X$. Then $\tilde{H}_0(X;G) \cong H_0(X,*;G)$.

Denote by $\pi_0(X)$ the set of path-connected components of X. For $c \in \pi_0(X)$ denote by $X_c \subset X$ the component corresponding to c $\ni$

$$H_0(X;G) \cong \bigoplus_{c \in \pi_0(X)} H_0(X_c;G) \cong \bigoplus_{c \in \pi_0(X)} G.$$

EXAMPLE

$$\tilde{H}_k(S^n;G) = \begin{cases} G & k=n \\ 0 & k \neq n \end{cases} \quad \text{if } n \geq 1.$$

For $n=0$, $H_k(S^0;G) = \begin{cases} 0 & k \neq 0 \\ G \oplus G & k=0 \end{cases}$.

Proof

For $n=0$, $S^0 = \{-1,1\}$ & the result follows from what we said before.

From now on omit G from the notation $H_*(-;G)$.

Assume $n \geq 1$. Consider $(B^n, \underset{S^{n-1}}{\underset{\wr\wr}{\partial B^n}})$

$(B, \partial B^n)$ is a good pair.

((X, A) is a good pair if $\emptyset \neq A \subset X$ is closed & $\exists$ nbhd $\mathcal{N}$ of A s.t. $A \subset \mathcal{N}$ is a strong deformation retract of $\mathcal{N}$)

Examples: X=CW complex, $\phi \neq A \subset X$ a subcomplex.

Then $H_k(B, \partial B^n) \cong \tilde{H}_k(B/\partial B^n)$. The same statement holds for homology with coefficients

We'll now use the LES of $(B^n, \partial B^n)$ & the theorem about good pairs & get

$$\cdots \to \overset{0}{\cancel{\tilde{H}_k(B^n)}} \to \underset{\approx S^n}{\tilde{H}_k(B^n/\partial B^n)} \xrightarrow{\cong} H_{k-1}(\underbrace{\partial B^n}_{S^{n-1}}) \to \overset{0}{\cancel{\tilde{H}_{k-1}(B^n)}} \to$$

$$\Rightarrow \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}).$$

$$\Rightarrow \tilde{H}_k(S^n) \cong \cdots \cong \tilde{H}_{k-n}(S^0) = \begin{cases} G & k=n \\ 0 & k \neq n \end{cases}$$

∎

How to see this iso more explicitly? What are the chain level representatives of the elements of $\tilde{H}_n(S^n;G)$?

Consider $\sigma_0 : \Delta^n \to \Delta^n/\partial\Delta^n$ the quotient map, viewed here as an n-dim simplex in the space $\Delta^n/\partial\Delta^n$ $(\approx S^n)$.

Note that σ_0 is an n-cycle in the group $S_n(\Delta^n/\partial\Delta^n, *)$, where $* \in \Delta^n/\partial\Delta^n$ corresponds to the points of $\partial\Delta^n$.

① One can show that $[\sigma_0] \in H_n(\Delta^n/\partial\Delta^n, *)$

$$\cong H_n(S^n, *) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}$$

is a generator (Example 2.23, page 125, Hatcher)

② Consider the following map

$$\begin{tikzcd} G \arrow[r] & H_n(\Delta^n/\partial\Delta^n, *; G) \\ g \arrow[u, phantom, "\in" rotate=-90] \arrow[r, mapsto] & {[g\sigma_0]} \arrow[u, phantom, "\in"] \end{tikzcd}$$

this is an iso.