THEOREM (POINCARE DUALITY)

Let M be a closed R-oriented n-manifold with fundamental class $[M] \in H_n(M;R)$ (corresponding to the given orientation). Then the map

$$PD: H^k(M;R) \longrightarrow H_{n-k}(M;R)$$
$$\alpha \longmapsto \alpha \cap [M]$$

is an isomorphism of R-modules for all k.

← previous theorem follows from this

The map $H^k(M;R) \to H_{n-k}(M;R)$ is called the cap product & it is what we take a look at next

CAP PRODUCT

For an arbitrary space X and

coefficient ring R, define an R-bilinear cap product

$$\cap : S_k(X;R) \times S^\ell(X;R) \to S_{k-\ell}(X;R)$$

for $k \geq \ell$ by setting:

$$\sigma \cap \varphi = \varphi(\sigma[e_0,\dots,e_\ell])\,\sigma[e_\ell,\dots,e_k]$$

for $\sigma : \Delta_k \to X$ and $\varphi \in S^\ell(X;R)$.

LEMMA

$$\partial(\sigma \cap \varphi) = (-1)^\ell(\partial\sigma \cap \varphi - \sigma \cap \delta\varphi)$$

PROOF OF LEMMA

$$\partial\sigma \cap \varphi = \sum_{i=0}^{\ell}(-1)^i\varphi(\sigma[e_0,\dots,\hat{e}_i,\dots,e_{\ell+1}])\,\sigma[e_{\ell+1},\dots,e_k]$$
$$+ \sum_{i=\ell+1}^{k}(-1)^i\varphi(\sigma[e_0,\dots,e_\ell])\,\sigma[e_\ell,\dots,\hat{e}_i,\dots,e_k]$$

$$\sigma \cap \delta\varphi = \sum_{i=0}^{\ell+1}(-1)^i\varphi(\sigma[e_0,\dots,\hat{e}_i,\dots,e_{\ell+1}])\,\sigma[e_{\ell+1},\dots,e_k]$$

$$\partial(\sigma\cap\varphi) = \sum_{i=\ell}^{k}(-1)^{i-\ell}\varphi(\sigma[e_0,\dots,e_\ell])\,\sigma[e_\ell,\dots,\hat{e}_i,\dots,e_k]$$

$$\Rightarrow (-1)^\ell(\partial\sigma\cap\varphi - \sigma\cap\delta\varphi) =$$

$$(-1)^{-\ell}\left(\sum_{i=\ell+1}^{k}(-1)^{i}\varphi(\sigma[e_0,\dots,\hat{e}_i,\dots,e_{\ell+1}])\,\sigma[e_{\ell+1},\dots,e_k]\right)=\partial(\sigma\cap\varphi)$$
$$-(-1)^{1}\varphi(\sigma[e_0,\dots,e_\ell])\,\sigma[e_{\ell+1},\dots,e_k]$$

∎

COROLLARY

cycles

(1) $\cap : Z_k \times Z^\ell \to Z_{k-\ell}$

(2) $\cap(B_k \times Z^\ell), \cap(Z_k \times B^\ell) \subseteq B_{k-\ell}$

(3) cap product on the level of chains/cochains induces cap product

$$\cap : H_k(X;R)\times H^\ell(X;R) \to H_{k-\ell}(X;R).$$

which is R-linear in every variable.

PROOF

(1) From the relation $\partial(\sigma\cap\varphi)=\pm(\partial\sigma\cap\varphi-\sigma\cap\delta\varphi)$ it follows that the cap product of a cycle σ and a cocycle φ is a cycle.

(2) If $\partial\sigma=0$ then $\partial(\sigma\cap\varphi)=\pm(\sigma\cap\delta\varphi)$, so the cap product of a cycle and a coboundary is a boundary.

If $\delta\varphi=0$, $\partial(\sigma\cap\varphi)=\pm\,\partial\sigma\cap\varphi$, so the cup product of a boundary and a cocycle is a boundary.

③ Is a consequence of ①&②. ∎

Relative forms of the cap product also exist:

$$H_k(X,A;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X,A;R)$$
$$H_k(X,A;R) \times H^\ell(X,A;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$$

PROPOSITION (NATURALITY OF $\frown$)

$\frown$ is natural w.r.t. maps in the sense that $\forall$ spaces X, Y, $X \xrightarrow{f} Y$, $\varphi \in H^\ell(Y)$, $\alpha \in H_k(X)$ we have

$$f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$$

$$\begin{array}{ccccc} H_k(X) & \times & H^\ell(X) & \xrightarrow{\frown} & H_{k-\ell}(X) \\ {\scriptstyle f_*}\downarrow & & \uparrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f_*} \\ H_k(Y) & \times & H^\ell(Y) & \xrightarrow{\frown} & H_{k-\ell}(Y) \end{array}.$$

Exercise.

EXAMPLE: SURFACES

For n-manifolds that have the structure of a Δ-complex, we have an explicit construction for a fundamental class. Consider the case with $\mathbb{Z}$-coefficients: In simplicial homology a fundamental class must be represented by some linear combination $\sum_i k_i \sigma_i$ of the n-simplices σ_i of M. Because the fundamental class maps to a generator of $H_n(M|x;\mathbb{Z})$ for points in the interior of the σ_i's each coefficient must be ± 1. The k_i's must also be such that $\sum k_i \sigma_i$ is a cycle. $\Rightarrow$ if σ_i & σ_j and σ_j share a common $(n-1)$-dim face, then k_i determines k_j & vice versa.

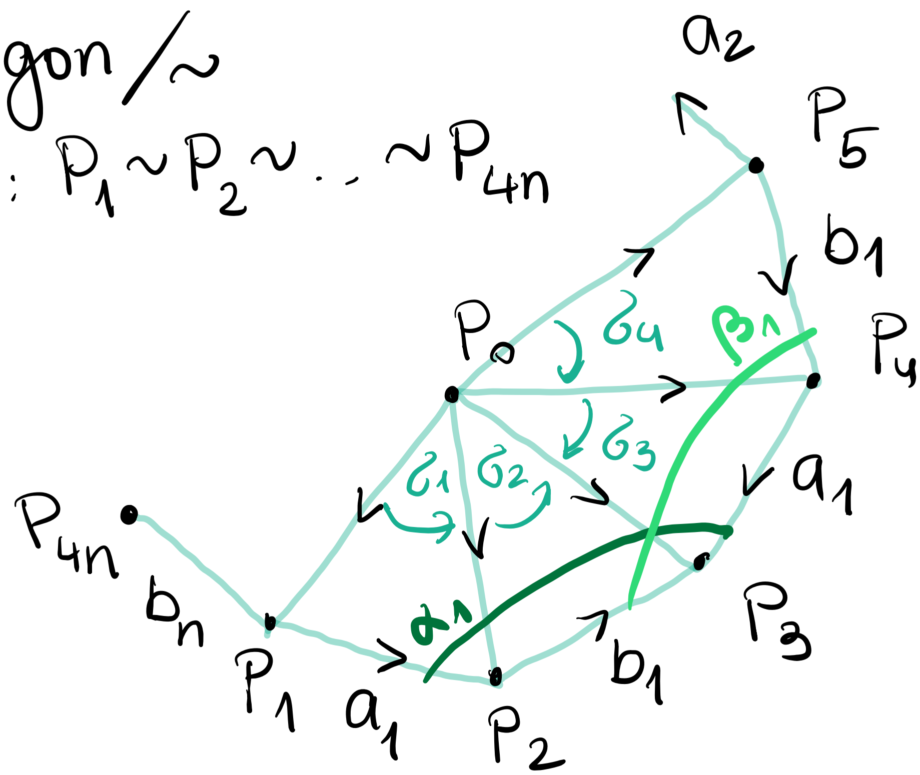
in $\mathbb{Z}_2$ $\sum \sigma_i$ always defines a fundamental class

ORIENTABLE SURFACES

$X = nT$

$X = 4n\text{-gon}/\sim$

vertices: $P_1 \sim P_2 \sim \dots \sim P_{4n}$



$$PD: H^1(X) \longrightarrow H_1(X) \leftarrow \begin{array}{ll} H^2(X) \to H_0(X) & c \cdot P \\ H^0(X) \to H_2(X) & c \cdot \tau \end{array}$$

$[X] = [\tau]$, $\tau = \sigma_1 + \sigma_2 - \sigma_3 - \sigma_4 + \dots$

$$\tau \cap \alpha_1 \overset{\text{linearity}}{=} \sigma_2 \cap \alpha_1 = \alpha_1(\sigma_2[P_0, P_2])\sigma_2[P_2, P_3] = 1 \cdot b_1$$

$$\tau \cap \beta_1 \underset{\text{linearity}}{=} -\sigma_3 \cap \beta_1 = -\beta_1(\sigma_3[P_0, P_4])\sigma_3[P_4, P_3] = -1 \cdot a_1$$

PD

$PD([\alpha_i]) = [b_i]$

$PD([\beta_i]) = -[a_i]$

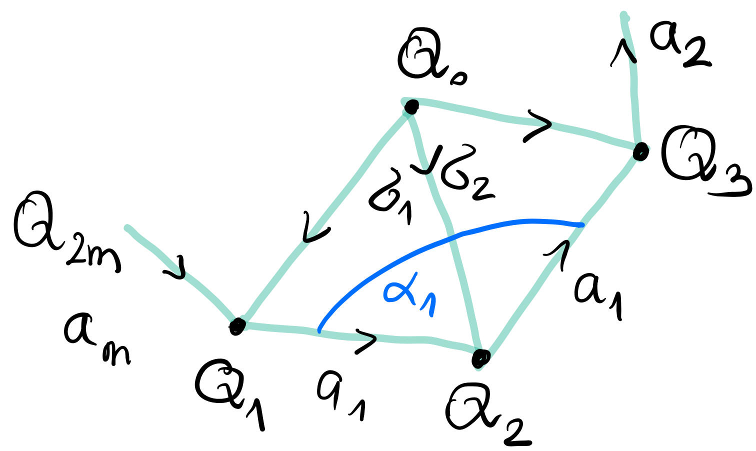
geometrically: the loops α_i & b_i are homotopic & so are the loops β_i & a_i

NON-ORIENTABLE SURFACES

($\mathbb{Z}_2$-orientable)

$X \approx mP$, $m \geq 1$

X admits a structure of Δ-complex with 2m 2-simplices, $X = 2m\text{-gon}/\sim$.



vertices: $Q_1 \sim Q_2 \sim \dots \sim Q_{2m}$

$\mathbb{Z}_2$-coefficients $\quad [\tau] = [\sigma_1 + \sigma_2 + \dots + \sigma_{2m}]$.

$$\tau \cap \alpha_1 = \sigma_2 \cap \alpha_1 = \alpha_1(\sigma_2[Q_0, Q_2])\,\sigma_2[Q_2, Q_3]$$
$$= 1 \cdot a_1$$

$\vdots$

$\tau \cap \alpha_i = 1 \cdot a_i \quad \rightsquigarrow a_i$ is the Poincaré dual of α_i (geometrically, the a_i loops are homotopic to α_i)

Comparing definitions of $\cap$ and $\cup$ we see that the composition

$$S^{\ell}(X)\times(S_{k+\ell}(X)\times S^{k}(X))\xrightarrow{id\times\cap}S^{k+\ell}(X)\times S_{k+\ell}(X)\xrightarrow{ev}R$$

is the same as

$$(S^{k}(X)\times S^{\ell}(X))\times S_{k+\ell}(X)\xrightarrow{\cup\times id}S^{k+\ell}(X)\times S_{k+\ell}(X)\xrightarrow{ev}R$$

so

$$\beta(\sigma\cap\alpha)=(\alpha\cup\beta)(\sigma).$$

Hence PD induces a pairing

$$H^{n-k}(X)\times H^{k}(X)\to R$$

$$(\beta,\alpha)\mapsto\beta([X]\cap\alpha)=(\alpha\cup\beta)[X].$$