COHOMOLOGY WITH COMPACT SUPPORTS

Let G be a group of coefficients.
$S^\circ(X;G)$. Define $S_c^\circ(X;G) \subset S^\circ(X;G)$ as follows.

$\varphi \in S^i(X;G)$ is called a

cochain with compact support if $\exists$ compact subset $K_\varphi \subset X$ s.t. $\varphi(\sigma) = 0$ $\forall$ chain σ in $X \setminus K_\varphi$.

$$S_c^i(X;G) = \{\text{compactly supported cochains in } S^i(X;G)\}$$

Note that if $\varphi \in S_c^i(X;G) \Rightarrow$ $\delta\varphi \in S_c^{i+1}(X;G)$ because $\delta\varphi(\sigma) = \varphi(\partial\sigma)$ and if σ is in $X \setminus K_\varphi$ then $\partial\sigma$ is in $X \setminus K_\varphi$. So

$S_c^\bullet \subset S^\bullet$ is a subcomplex.

We write $H_c^i(X;G) := H^i(S_c^\bullet(X;G))$.

We call it COHOMOLOGY WITH COMPACT SUPPORT.

Interpretation in terms of direct limits

Let $\{G_\alpha\}_{\alpha \in I}$ be a collection of abelian groups indexed by a directed set I (this means I is partially ordered and $\forall \alpha, \beta \in I$ $\exists \gamma \in I$ s.t. $\gamma \geq \alpha$ and $\gamma \geq \beta$).

Suppose we are also given $\forall \alpha \leq \beta$ in I a homomorphism

$$f_{\beta\alpha} : G_\alpha \to G_\beta \quad \text{s.t.}$$

$f_{\alpha\alpha} = id$ $\forall \alpha$, If $\alpha \leq \beta \leq \gamma$ then

$f_{\gamma\alpha} = f_{\gamma\beta} \circ f_{\beta\alpha}$. We call such a structure a directed system of groups. Define a group $\varinjlim_{\alpha \in I} G_\alpha$ as follows:

Consider $\bigsqcup_{\alpha \in I} G_\alpha$. Define an equivalence relation: $a \in G_\alpha$, $b \in G_\beta$ are declared equivalent $a \sim b$ iff $\exists\, \gamma \geq \alpha, \beta$ s.t.

$$f_{\gamma\alpha}(a) = f_{\gamma\beta}(b).$$

colimit in category theory

$$\lim_{\alpha \in I} G_\alpha := \bigsqcup_{\alpha \in I} G_\alpha \Big/ \sim$$

CLAIM

$\varinjlim_{\alpha \in I} G_\alpha$ is an abelian group. If $a \in G_\alpha$, $b \in G_\beta$, then $[a] + [b] := [a' + b']$, where $a' = f_{\gamma\alpha}(a)$, $b' = f_{\gamma\beta}(b)$ for some $\mu \geq \alpha, \beta$.

Exercise: Check the details.

REMARK: If $J \subset I$ is a subset with the property that $\forall \alpha \in I, \exists \beta \in J$ with $\beta \geq \alpha$, then

$$\varinjlim_{\alpha \in J} G_\alpha \xrightarrow{\cong} \varinjlim_{\alpha \in I} G_\alpha$$ is

an iso. In particular, if I has a maximal element μ (i.e. $\mu \geq \alpha \ \forall \alpha \in I$), then $\varinjlim_{\alpha \in I} G_\alpha \cong G_\mu$.

The inclusions $G_\alpha \to \bigoplus_{\mu \in I} G_\mu$ induce homomorphisms $i_\alpha : G_\alpha \to \varinjlim_{\mu \in I} G_\mu$

& $\forall \beta \geq \alpha$ we have

$$\begin{tikzcd}
G_\alpha \arrow[dr, "i_\alpha"] \arrow[d, "f_{\beta\alpha}"'] & \\
G_\beta \arrow[r, "i_\beta"'] \arrow[ur, phantom, "\circlearrowright"] & \varinjlim_{\mu \in I} G_\mu .
\end{tikzcd}$$

Note also that $\forall g \in \lim_{\mu \in I} G_\alpha, \exists \alpha \in I$, $g_\alpha \in G_\alpha$ s.t. $g = i_\alpha(g_\alpha)$.

PROPOSITION

Let $\{G_\alpha\}_{\alpha \in I}$, $\{f_{\beta\alpha}\}_{\beta \geq \alpha}$ be a directed system of abelian groups. Let A be an abelian group, and $h_\alpha : G_\alpha \to A$, $\alpha \in I$, homomorphisms s.t. $\forall \beta \geq \alpha$, $h_\beta \circ f_{\beta\alpha} = h_\alpha$. Then $\exists!$ homo. $h : \varinjlim_{\alpha \in I} G_\alpha \to A$ s.t. $h \circ i_\alpha = h_\alpha$ $\forall \alpha \in I$.

$$\begin{tikzcd}
G_\alpha \arrow[dr, "i_\alpha"] \arrow[drr, "h_\alpha"] \arrow[dd, "f_{\beta\alpha}"'] & & \\
& \varinjlim_{\mu \in I} G_\mu \arrow[r, dashed, "\exists h"] & A \\
G_\beta \arrow[ur, "i_\beta"] \arrow[urr, "h_\beta"'] & &
\end{tikzcd}$$

Moreover, $\mathrm{im}(h) = \bigcup_{\alpha \in I} \mathrm{im}(h_\alpha)$ and $\ker(h) = \bigcup_{\alpha \in I} i_\alpha(\ker h_\alpha)$.

COROLLARY

$h : \varinjlim_{\alpha \in I} G_\alpha \to A$ is an iso iff the following two things hold:

① $\forall a \in A$, $\exists \alpha \in I$, $g_\alpha \in G_\alpha$ s.t. $h_\alpha(g_\alpha) = a$.

② If $h_\alpha(g_\alpha) = 0$, then $\exists \beta \geq \alpha$ s.t. $f_{\beta\alpha}(g_\alpha) = 0$.

PROPOSITION (EXACTNESS)

Let $\{G_\alpha'\}, \{G_\alpha\}, \{G_\alpha''\}$, $\alpha \in I$ be directed systems of abelian groups. Suppose $\forall \alpha \in I$ we have an exact sequence

$$G_\alpha' \xrightarrow{\varphi_\alpha} G_\alpha \xrightarrow{\Psi} G_\alpha''$$

and that $\forall \beta \geq \alpha$ this diagram is commutative

$$\begin{tikzcd}
G_\alpha' \arrow[r, "\varphi_\alpha"] \arrow[d, "f'_{\beta\alpha}"'] & G_\alpha \arrow[r, "\Psi_\alpha"] \arrow[d, "f_{\beta\alpha}"] & G_\alpha'' \arrow[d, "f''_{\beta\alpha}"] \\
G_\beta' \arrow[r, "\varphi_\beta"'] & G_\beta \arrow[r, "\Psi_\beta"'] & G_\beta''
\end{tikzcd}$$

then $\varinjlim_{\alpha \in I} G_\alpha' \to \varinjlim_{\alpha\in I} G_\alpha \to \varinjlim_{\alpha \in I} G_\alpha''$

is also exact.

Exercise

Back to topology.

Suppose $X = \bigcup_{\alpha \in I} X_\alpha$ and that $\forall \beta \geq \alpha$

we have $X_\beta \supset X_\alpha$. Then the groups

$\{H_i(X_\alpha; G)\}_{\alpha \in I}$ together with

$$f_{\beta\alpha} := \left(H_i(X_\alpha; G) \xrightarrow{inc_*} H_i(X_\beta; G)\right)$$

form a directed system.

Moreover, the maps $H_i(X_\alpha;G) \xrightarrow{inc_*} H_i(X;G)$ induce a homo.

$$\varinjlim_{\alpha\in I} H_i(X_\alpha;G) \to H_i(X;G)$$

COROLLARY

Suppose $X = \bigcup_{\alpha\in I} X_\alpha$ as above & suppose that $\forall$ compact subset $K \subset X$, $\exists \alpha \in I$ s.t. $X_\alpha \supset K$. Then

$$\lim_{\alpha\in I} H_i(X_\alpha;G) \to H_i(X;G)$$

is an iso $\forall i$.

Let X be a space. The compact subsets $K \subset X$ form a directed set (wrt inclusion), because if $K_1, K_2 \subset X$ are compact, then $K_1 \cup K_2$ is also compact. $\forall$ compact subset $K \subset X$

we associate

$$H^i(X|K;G) := H^i(X, X\setminus K; G).$$

If $K \subset L \subset X$ (with K, L compact), we have the homomorphism:

$$H^i(X|K;G) \xrightarrow{R_{L,K}} H^i(X|L;G)$$

induced by $(X, X\setminus L) \xhookrightarrow{inc} (X, X\setminus K)$.

CLAIM

$$H^i_c(X;G) \cong \varinjlim_{K \subset X} H^i(X|K;G).$$

PROOF

$\forall$ compact subset $K \subset X$ we have an obvious homomorphism

$$H^i(X|K;G) \xrightarrow{h_K} H^i_c(X;G)$$

defined as follows:

$$\overline{h}_K : S^i(X, X\setminus K; G) \to S^i_c(X;G)$$

Let $\varphi \in S^i(X, X\setminus K; G)$, ie. $\varphi : \frac{S_i(X)}{S_i(X\setminus K)} \to G$, then

$$\tilde{\varphi} := \left(S_i(X) \to \frac{S_i(X)}{S_i(X\setminus K)} \xrightarrow{\varphi} G \right)$$

is a cochain with compact support.

$\tilde{\varphi}(\sigma) = 0$ for every chain $\sigma \subset X \setminus K$.

Define $\overline{h_K}(\varphi) := \tilde{\varphi} \in S^i_c(X; G)$.

Clearly, this is a chain map.

$\Rightarrow$ it induces a map

$$h_K : H^i(X|K; G) \to H^i_c(X; G).$$

Now $h_L \circ R_{L,K} = h_K \; \forall \, K \subset L$ compact.

$\Rightarrow$ we get $h : \varinjlim_{K \subset X} H^i(X|K; G) \to H^i_c(X; G)$.

Denote by $i_K : H^i(X|K; G) \to \varinjlim_{K \subset X} H^i(X|K; G)$

the maps that come with the construction of direct lim.

We'll show h is injective. Let $a \in \ker h_K$.

Suppose $a = [\varphi]$. $h_K(a) = [\tilde{\varphi}]$, where

$$\tilde{\varphi} = \left(S_i(x) \to S_i(x) / S_i(X \setminus K) \xrightarrow{\varphi} G \right).$$

Since $h_K(a) = 0$ $\exists$ cochain $\Psi : S_{i-1}(X) \to G$

with support in some compact

$K' \subset X$ s.t. $\tilde{\varphi} = \Psi \circ \partial$. Clearly, $i_K(a) = 0$

because $R_{K \cup K', K}([\varphi]) = 0$. $\Rightarrow$

$i_K(\ker h_K) = 0$. $\Rightarrow \ker(h) = 0 \Rightarrow$

h is injective.

We'll show now that h is surjective.

Recall $\operatorname{im}(h) = \bigcup_{K \subset X} \operatorname{im}(h_K)$.

Let $b \in H^i_c(X; G)$ and $\varphi : S_i(X) \to G$,

$b = [\varphi]$. Assume φ is supported

in the compact subset $K \subset X$.

$\Rightarrow \varphi_{S_i(X\setminus K)} \equiv 0.$

$\Rightarrow \varphi$ induces $\bar{\varphi} : S_i(X) / S_i(X \setminus K) \to G$

and $\bar{h}_K(\bar{\varphi}) = \varphi. \Rightarrow b = h_K([\bar{\varphi}]).$ $\blacksquare$

EXAMPLE

$H_c^*(\mathbb{R}^n; G).$

$$H_c^i(\mathbb{R}^n; G) \cong \varinjlim_{K \subset \mathbb{R}^n} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K; G)$$

$$\cong \varinjlim_{B(R)} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B(R); G)$$

But

$$H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B(R); G) \cong \begin{cases} G \\ 0 \end{cases}$$

(follows from LES for $(\mathbb{R}^n, \mathbb{R}^n \setminus B(R))$).

$\forall\ R_1 < R_2$ we have that

$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B(R_1); G) \to H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B(R_2); G)$ is iso.

$$\Rightarrow H_c^i(\mathbb{R}^n; G) \cong \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}$$

Compare: $H^i(\mathbb{R}^n; G) = \begin{cases} 0 & i \neq 0 \\ G & i = 0 \end{cases}$

$$H_i(\mathbb{R}^n; G) = \begin{cases} 0 & i \neq 0 \\ G & i = 0 \end{cases}$$