Let M be an R-orientable n-mnfd, not necessarily compact. Fix an R-orientation μ on M. Define

$$PD : H^k_c(M;R) \to H_{n-k}(M;R).$$

$\forall\, K \subset L \subset M$ compact subsets we have

$$\begin{tikzcd}
H^k(M|L;R) \times H_n(M|L;R) \arrow[dr, "\frown"] \arrow[dd, shift left=2, "(L_{L,K})"', "i_{K,L}"] & \\
 & H_{n-k}(M;R) \\
H^k(M|K;R) \times H_n(M|K;R) \arrow[ur, "\frown"'] \arrow[uu, shift left=2, "R_{L,K}"] &
\end{tikzcd}$$

By a previous lemma $\exists!\, \mu_K \in H_n(M|K)$, $\mu_L \in H_n(M|L)$ s.t. $L_{K,x}(\mu_K) = \mu_x$ $\forall x \in K$ & $L_{L,x}(\mu_L) = \mu_x$ $\forall x \in L$.

By the uniqueness of μ_K & μ_L we have $i_{K,L}(\mu_L) = \mu_K$.

By naturality of the cup product we have: $\alpha \cap i_*(\mu_L) = i^*\alpha \cap \mu_L$

$(M, M\setminus L) \xrightarrow{i} (M, M\setminus K)$ is the inclusion.

$$i_* = i_{K,L}, \quad i^* = R_{L,K}.$$

So we get $\alpha \cap \mu_K = R_{L,K}(\alpha) \cap \mu_L$.

$\Rightarrow$ the homomorphisms

$$H^k(M|K; R) \longrightarrow H_{n-k}(M; R)$$
$$\alpha \longmapsto \alpha \cap \mu_K$$

induce a map $\underbrace{\varinjlim_{K \subset M} H^k(M|K; R)}_{\cong\, H^k_c(M; R)} \to H_{n-k}(M; R)$

We denote this map $PD : H^k_c(M; R) \to H_{n-k}(M; R)$.

For the proof of PD we also need the following lemma:

LEMMA

Suppose M is an R-oriented n-manifold and $M = \mathcal{U} \cup \mathcal{V}$, $\mathcal{U}, \mathcal{V}$ = open. Then $\exists$ a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to & H_c^k(\mathcal{U}\cap\mathcal{V}) & \to & H_c^k(\mathcal{U}) \oplus H_c^k(\mathcal{V}) & \to & H_c^k(M) & \to & H_c^{k+1}(\mathcal{U}\cap\mathcal{V}) & \to \cdots \\
& \downarrow {\scriptstyle PD_{\mathcal{U}\cap\mathcal{V}}} & & \downarrow {\scriptstyle PD_{\mathcal{U}} \oplus PD_{\mathcal{V}}} & & \downarrow {\scriptstyle PD_M} & & \downarrow {\scriptstyle PD_{\mathcal{U}\cap\mathcal{V}}} & \\
\cdots \to & H_{n-k}(\mathcal{U}\cap\mathcal{V}) & \to & H_{n-k}(\mathcal{U}) \oplus H_{n-k}(\mathcal{V}) & \to & H_{n-k}(M) & \to & H_{n-k-1}(\mathcal{U}\cap\mathcal{V}) & \to \cdots
\end{array}$$

the rows are MV type of LESs. All coefficients are in R.

PROOF

Recall the relative MV: $(X, Y) = (A \cup B, C \cup D)$ with $C \subset A$, $D \subset B$ and s.t. $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, $Y = \mathrm{Int}(C) \cup \mathrm{Int}(D)$, then

$$\cdots \to H^k(X,Y) \xrightarrow{\Psi} H^k(A,C)\oplus H^k(B,D) \xrightarrow{\varphi} H^k(A\cap B, C\cap D) \to \cdots$$

$$\Psi(\alpha) = \left(\alpha\big|_{S.(A)}, \alpha\big|_{S.(B)}\right)$$

$$\varphi(\beta,\eta) = \beta\big|_{S.(A\cap B)} - \eta_{S.(A\cap B)}$$

We'll use this with $A = B = M$, $C = M\setminus K$, $D = M\setminus L$, where $K\subset \mathcal{U}$, $L\subset V$ are compact.

We get the 1st row of the following diag.

$$\begin{tikzcd}
\to H^k(M|K\cap L) \arrow[r] \arrow[d, "\cong"', "\text{exc.}"] \arrow[dr, phantom, "c"] & H^k(M|K)\oplus H^k(M|L) \arrow[r] \arrow[d, "\text{exc.}"', "\cong"] \arrow[dr, phantom, "2"] & H^k(M|K\cup L) \arrow[r, "c'"] \arrow[dd, "(-)\cap\mu_{K\cup L}"'] \arrow[dr, phantom, "3"] & H^{k+1}(M|K\cap L) \arrow[d, "\cong"', "\text{exc}"] \\
H^k(U\cap V, K\cap L) \arrow[r] \arrow[d, "(-)\cap\mu_{K\cap L}"] \arrow[dr, phantom, "1"] & H^k(\mathcal{U}|K)\oplus H^k(\mathcal{V}|L) \arrow[d, "(-)\cap\mu_K\oplus(-)\cap\mu_L"] & & H^{k+1}(U\cap V, K\cap L) \arrow[d, "(-)\cap\mu_{K\cap L}"'] \\
\to H_{n-k}(\mathcal{U}\cap V) \arrow[r] & H_{n-k}(\mathcal{U})\oplus H_{n-k}(\mathcal{V}) \arrow[r] & H_{n-k}(M) \arrow[r, "c''"] & H_{n-k-1}(\mathcal{U}\cap\mathcal{V}) \to
\end{tikzcd}$$

The bottom row is homological M-V.

The vertical maps from the orientation, ie. $\mu_{K\cap L}\in H_n(\mathcal{U}\cap\mathcal{V}|K\cap L)$, $\mu_L\in H_n(V|L)$,

$\mu_K \in H_n(\mathcal{U}|K)$, $\mu_{K\cup L} \in H_n(\mathcal{U}\cup\mathcal{V}|K\cup L)$ are the restrictions of the given orientation to $K\cap L$, K, L, etc.

CLAIM

Squares ①, ②, ③ are commutative, hence the diagram commutes.

Squares ① & ② commute on the chain/cochain level.

$$\begin{tikzcd}
H^k(M|K\cup L) \arrow[r, "c'"] \arrow[ddd, "(-)\cap\mu_{K\cup L}"] & H^{k+1}(M|K\cap L) \arrow[d, "\cong"] \\
& H^{k+1}(\mathcal{U}\cup\mathcal{V}|K\cap L) \arrow[dd, "(-)\cap\mu_{K\cap L}"] \\
& \\
H_{n-k}(M) \arrow[r, "c''"] & H_{n-k-1}(\mathcal{U}\cap\mathcal{V})
\end{tikzcd} \qquad (*)$$

The map c': Put $C = M\setminus K$, $D = M\setminus L$. Let $S_\cdot^{C,D} \subset S_\cdot(C\cup D)$ be the subcomplex generated by the chains in C & the chains in D.

$$\mathcal{D} := \frac{S_\cdot(M)}{S_\cdot^{C,D}}, \quad \mathcal{D}^* := \hom(\mathcal{D}, R).$$

$\mathcal{D}^*$ = cochains in M that vanish on the chains in C and on the chains in D.

Recall $S_\cdot^{C,D} \xrightarrow{\text{inc.}} S_\cdot(C \cup D)$ induces an isomorphism on homology. $\Rightarrow$

$$\mathcal{D}^* \leftarrow S^\cdot(M, C\cup D) = S^\cdot(M, M \setminus (K \cap L)).$$

$$0 \to \mathcal{D}^* \xrightarrow{\Psi} S^\cdot(M,C) \oplus S^\cdot(M,D) \xrightarrow{\varphi} S^\cdot(M, C\cap D) \to 0$$

$$a \xmapsto{\Psi} (a,a)$$

$$(b,c) \xmapsto{\varphi} b-c$$

REMARK

From this sequence & the fact that $H^*(S^\cdot(M, C\cup D)) \xrightarrow{\cong} H^*(\mathcal{D}^*)$ we get the sequence on the top, from the beginning of the proof.

How to calculate $c'([\alpha])$ for a cocycle $\alpha \in S^\cdot(M, C \cap D)$.

1st step

$$\alpha = \alpha_C - \alpha_D \text{ with } \alpha_C \in S^\cdot(M, C), \alpha_D \in S^\cdot(M, D).$$

Note that $\delta\alpha_C - \delta\alpha_D = \delta\alpha = 0 \Rightarrow$

$$\delta\alpha_C = \delta\alpha_D.$$

2nd step

$$(\delta\alpha_C, \delta\alpha_D) = \Psi(\mu), \quad \mu = \delta\alpha_C = \delta\alpha_D.$$

$$c'([\alpha]) = [\delta\alpha_C] \in H^{k+1}(\mathcal{D}^*) \cong H^{k+1}(M \mid K \cap L)$$

this is not necessarily a coboundary in $\mathcal{D}^*$ bc α_C might <u>not</u> belong to $\mathcal{D}^*$ ($\delta\alpha_C$ is in $\mathcal{D}^*$)

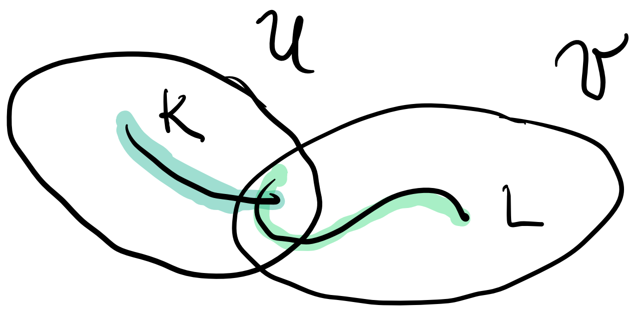
We need to calculate $c'([\alpha]) \cap \mu_{K \cap L}$.

Consider the class $\mu_{K \cup L} \in H_n(M \mid K \cup L)$.

The open sets $\mathcal{U} \setminus L, \mathcal{U} \cap \mathcal{V}, \mathcal{V} \setminus K$

cover $M = \mathcal{U} \cup \mathcal{V}$ (because $(\mathcal{U} \setminus L) \cup (\mathcal{U} \cap \mathcal{V}) = \mathcal{U}$,

$(\mathcal{V} \setminus K) \cup (\mathcal{U} \cap \mathcal{V}) = \mathcal{V}$)



Using standard Mayer-Vietoris arguments (barycentric subdivision etc) arguments we can represent $\mu_{K \cup L}$ by a chain

$$X = X_{\mathcal{U} \setminus L} + X_{\mathcal{U} \cap \mathcal{V}} + X_{\mathcal{V} \setminus K}$$

$$\begin{matrix} \cap & \cap & \cap \\ S_n(\mathcal{U} \setminus L) & S_n(\mathcal{U} \cap \mathcal{V}) & S_n(\mathcal{V} \setminus K) \end{matrix}$$

Consider now $\mu_{K \cap L} = L_{K \cup L, K \cap L}(\mu_{K \cup L}) =$

$$= [X] \in H_n(M, M \setminus (K \cap L)).$$

But in $S_n(M) / S_n(M \setminus (K \cap L))$ we have $x_{U \setminus L} = 0$, $x_{V \setminus K} = 0$ (because
$U \setminus L \subset M \setminus (K \cap L)$
$V \setminus L \subset M \setminus (K \cap L)$) $\Rightarrow \mu_{K \cap L} = [x_{U \cap V}]$.

In a similar way $\mu_K \in H_n(M \mid K)$ can be written as $\mu_K = [x_{U \setminus L} + x_{U \cap V}]$, $\mu_L = [x_{U \cap V} + x_{V \setminus K}]$. Let $\alpha \in S^k(M, M \setminus (K \cap L))$ be a cocycle. We've seen that $c^1[\alpha] = [\delta \alpha_c]$. So we need to calculate

$\delta \alpha_c \cap x_{U \cap V}$.

CLAIM

α_c might not be in $S^{\cdot}(U \cap V \mid K \cap L)$

$$[\delta \alpha_c \cap x_{U \cap V}] = (-1)^{k+1} [\alpha_c \cap \partial x_{U \cap V}]$$

the result might not be a boundary in $S_{\cdot}(U \cap V)$

PROOF

$$\partial(\alpha_c \cap x_{\mathcal{U}\cap\mathcal{V}}) = \delta\alpha_c \cap x_{\mathcal{U}\cap\mathcal{V}} + (-1)^k \alpha_c \cap \partial x_{\mathcal{U}\cap\mathcal{V}}$$

Now $\alpha_c \cap x_{\mathcal{U}\cap\mathcal{V}} \in S_\bullet(\mathcal{U}\cap\mathcal{V}) \Rightarrow$

$$[\delta\alpha_c \cap x_{\mathcal{U}\cap\mathcal{V}}] = (-1)^{k+1}[\alpha_c \cap \partial x_{\mathcal{U}\cap\mathcal{V}}] \in H_{n-k-1}(\mathcal{U}\cap\mathcal{V})$$

SUMMARY

$$\begin{tikzcd}
{[\alpha]} \arrow[r, mapsto, "c'"] & {[\delta\alpha_c]} \arrow[d, "\cap \mu_{K\cap L}"] \\
 & (-1)^{k+1}[\alpha_c \cap \partial x_{\mathcal{U}\cap\mathcal{V}}]
\end{tikzcd}$$

Consider now the other composition

$$\begin{tikzcd}
{[\alpha]} \arrow[d, mapsto] & \\
{[\alpha \cap x]} \arrow[r, mapsto, "c''"] & ?
\end{tikzcd}$$

$$\alpha \cap x = (\underbrace{\alpha \cap x_{\mathcal{U}\setminus L}}_{\text{chain in } \mathcal{U}}) + (\underbrace{\alpha \cap x_{\mathcal{U}\cap\mathcal{V}} + \alpha \cap x_{\mathcal{V}\setminus K}}_{\text{chains in } \mathcal{V}})$$

Exercise: $c''([\alpha \cap x]) = [\partial(\alpha \cap x_{\mathcal{U}\setminus L})]$

To finish the proof of the commutativity of 3 we need to show that

$$(-1)^{k+1}[\alpha_C \cap \partial x_{U\cap V}] = [\partial(\alpha \cap x_{U\setminus L})].$$

Indeed, $\partial(\alpha \cap x_{U\setminus L}) = \delta\alpha \cap x_{U\setminus L} + (-1)^k \alpha \cap \partial x_{U\setminus L}$

$= (-1)^k (\alpha_C - \alpha_D) \cap \partial x_{U\setminus L} =$

$= (-1)^k \alpha_C \cap \partial x_{U\setminus L} - (-1)^k \underbrace{\alpha_D \cap \partial x_{U\setminus L}}_{\substack{\| \\ 0 \text{ (because } \alpha_D \in S^k(M,D),\ D = M\setminus L, \\ \text{so } \alpha_D|_{S_\cdot(U\setminus L)} \equiv 0)}}$

$= (-1)^k \alpha_C \cap \partial x_{U\setminus L}.$ (**)

It remains to show:

$(-1)^{k+1}[\alpha_C \cap \partial x_{U\cap V}] = (-1)^k[\alpha_C \cap \partial x_{U\setminus L}] \in H_{n-k-1}(U\cap V)$

Note $\mu_K = [x_{U\cap V} + x_{U\setminus L}] \in H_n(M, M\setminus K) \Rightarrow$

$$\partial x_{U\cap V} + \partial x_{U\setminus L} \in S_{n-1}(\underbrace{M\setminus K}_{\supseteq C})$$

$$\Rightarrow \alpha_c \cap (\partial x_{u\cap v} + \partial x_{u \setminus L}) = 0, \text{ because}$$

$$\alpha_c\big|_{S.(c)} \equiv 0.$$

From $(**)$ we get

$$[\alpha_c \cap \partial x_{u\cap v}] = -[\alpha_c \cap \partial x_{u\setminus L}] = (-1)^{k+1}[\partial(\alpha \cap x_{u\setminus L})]$$

this completes the proof of 3.

Recall:

Let $\{G_\alpha'\}, \{G_\alpha\}, \{G_\alpha''\}$, $\alpha \in I$ be directed systems of graphs of abelian groups. Suppose $\forall \alpha \in I$ we have an exact sequence

$$G_\alpha' \xrightarrow{\varphi_\alpha} G_\alpha \xrightarrow{\Psi} G_\alpha''$$

and that $\forall \beta \geq \alpha$ this diagram is commutative

$$\begin{tikzcd}
G_\alpha' \arrow[r, "\varphi_\alpha"] \arrow[d, "f_{\beta\alpha}'"'] & G_\alpha \arrow[r, "\Psi_\alpha"] \arrow[d, "f_{\beta\alpha}"] & G_\alpha'' \arrow[d, "f_{\beta\alpha}''"] \\
G_\beta' \arrow[r, "\varphi_\beta"'] & G_\beta \arrow[r, "\Psi_\beta"'] & G_\beta''
\end{tikzcd}$$

Then $\varinjlim_{\alpha \in I} G'_\alpha \to \varinjlim_{\alpha \in I} G_\alpha \to \varinjlim_{\alpha \in I} G''_\alpha$

is also exact.

We'll apply this to the diagram

$$\begin{array}{ccccccc} \cdots \to H^k(U \cap V \mid K \cap L) & \to & H^k(U \mid K) \oplus H^k(V \mid L) & \to & H^k(M \mid K \cup L) & \to \\ \downarrow & & \downarrow & & \downarrow & \\ \cdots \to H_{n-k}(U \cap V) & \to & H_{n-k}(U) \oplus H_{n-k}(V) & \to & H_{n-k}(M) & \to \cdots \end{array}$$

Apply $\varinjlim_{(K,L)}$ where $K \subset U$, $L \subset V$ are compact

& $(K', L') \leq (K'', L'')$ iff $K' \subset K''$, $L' \subset L''$.

CLAIM

$\varinjlim_{(K,L)} H^k(M \mid K \cup L) \cong \varinjlim_{\substack{A \subset M \\ \text{compact}}} H^k(M \mid A)$.

Proof of the claim

$\forall A \subset M$ compact, $\exists K \subset U$, $L \subset V$ compact.

s.t. $A \subset K \cup L$. Just cover $A \cap \mathcal{U}$ by open balls $\bigcup_\alpha B'_\alpha$ with $\overline{B'_\alpha} \subset \mathcal{U}$ and cover $A \cap \mathcal{V}$ by open balls $\bigcup_\beta B''_\beta$ with $\overline{B''_\beta} \subset \mathcal{V}$.

Now take a finite subcovering of $\bigcup_\alpha \overline{B'_\alpha} \cup \bigcup_\beta \overline{B''_\beta}$ that covers A. This proves the claim.

CONCLUSION $\varinjlim_{(K,L)} H^k(M \mid K \cup L) \cong H^k_c(M)$.

Finally, $\varinjlim_{(K,L)} H^k(\mathcal{U} \cap \mathcal{V} \mid K \cap L) \cong \varinjlim_{\substack{B \subset \mathcal{U} \cap \mathcal{V} \\ \text{compact}}} H^k(\mathcal{U} \cap \mathcal{V})$

$$\cong H^k_c(\mathcal{U} \cap \mathcal{V}).$$

This proves the lemma.

$\blacksquare$

Now we finally prove the Poincaré Duality.

THEOREM (POINCARE DUALITY)

Let M be a closed R-oriented n-manifold with fundamental class $[M] \in H_n(M;R)$ (corresponding to the given orientation). Then the map

$$PD: H^k(M;R) \longrightarrow H_{n-k}(M;R)$$
$$\alpha \longmapsto \alpha \cap [M]$$

is an isomorphism of R-modules for all k.

PROOF

Claim 1 If $M = U \cup V$ and if PD_U, PD_V and $PD_{U \cap V}$ are all isos then PD_M is also an iso.

This follows from the previous lemma and the 5-lemma. ∎

Claim 2 Suppose I is a directed set and $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ are open subsets of M s.t. $\alpha\le\beta \Rightarrow U_\alpha \subset U_\beta$. Assume also that $\bigcup_{\alpha\in I} \mathcal{U}_\alpha = M$. If $PD_{\mathcal{U}_\alpha}$ is an iso for all α, then PD_M is an iso.

$$H_c^k(\mathcal{U}_\alpha) \cong \varinjlim_{\substack{K\subset \mathcal{U}_\alpha \\ \text{compact}}} H^k(M|K) \overset{\text{excision}}{\cong} H^k(\mathcal{U}_\alpha|K)$$

Note that if $\alpha\le\beta$ we have $H_c^k(\mathcal{U}_\alpha)\to H_c^k(\mathcal{U}_\beta)$ (because $S_c^k(\mathcal{U}_\alpha)\hookrightarrow S_c^k(\mathcal{U}_\beta)$: if $K\subset\mathcal{U}_\alpha$ is compact, $K\subset\mathcal{U}_\alpha$ is compact too).

Now $\varinjlim_{\alpha\in I} H_c^k(\mathcal{U}_\alpha) = \varinjlim_{\alpha\in I} \varinjlim_{\substack{K\subset\mathcal{U}_\alpha \\ \text{compact}}} H^k(\mathcal{U}_\alpha|K) \overset{*}{\cong}$

$\cong \varinjlim_{\substack{K\subset M \\ \text{compact}}} H^k(M|K) \cong H_c^k(M)$

* compact sets in M are just compact sets in all $\mathcal{U}_\alpha$

$$\Rightarrow \begin{tikzcd} \varinjlim_{\alpha \in I} H_c^k(U_\alpha) \arrow[r, "\cong"] \arrow[d, "\cong" left, "PD_{U_\alpha}"] & H_c^k(M) \arrow[d, "PD_M"] \\ \varinjlim_{\alpha \in I} H_{n-k}(U_\alpha) \arrow[r, "\cong"] & H_{n-k}(M) \end{tikzcd} \Rightarrow \begin{array}{l} PD_M \text{ is} \\ \text{an} \\ \text{iso.} \end{array}$$

every compact subset in M must be contained in some U_α: we had a lemma saying that in such a case $\varinjlim_{\alpha} H_*(U_\alpha) \cong H_*(M)$ ■

Step 1 $M = \mathbb{R}^n$.

Given a closed ball B we know that $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) = \mathbb{Z}$ with generator μ. By the UCT $h: H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B; \mathbb{Z}) \to \mathrm{Hom}(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B); \mathbb{Z})$ is an isomorphism. Then $\exists$ a generator a s.t. $a(\mu_B) = 1$. then

$1 = a(\mu_B) = (1 \cup a)(\mu_B) = 1(a \cap \mu_B) \Rightarrow$

$a \cap \mu_B$ is a generator of $H_0(\mathbb{R}^n) = \mathbb{Z}$.

Thus μ_B gives an isomorphism

$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \to H_0(\mathbb{R}^n)$ for all B.

Hence by universal property of direct limit, the map $PD_{\mathbb{R}^n}$ is an isomorphism in the case $i=n$. The cases $i \neq n$ is obvious since it maps 0 to 0.

Step 2 Let $M \subset \mathbb{R}^n$, and assume $M = \bigcup_{i\in I} \mathcal{U}_i$ with I finite & all U_i convex, open.

By step 1 $PD_{\mathcal{U}_i}$ is an iso because $\mathcal{U}_i \approx \mathbb{R}^n$. Now use induction on $|I|$:

Suppose $I = \{1, \dots, k\}$, put

$$V_2 := U_1 \cup \dots \cup U_{2-1}.$$

By induction PD is an iso for V_2 & $V_2 \cap U_2$ (and of course U_2 too).

$$\overset{\shortparallel}{(\mathcal{U}_1 \cap \mathcal{U}_2) \cup \dots \cup (\mathcal{U}_{2-1} \cap \mathcal{U}_2)}$$

convex

Since both V_g & $V_g \cap U_g$ are unions of at most $g-1$ open convex subsets $\Rightarrow$ PD is an iso also for $V_{g+1} = V_g \cup U_g$.

follows from Claim 1

Step 3 $M = \bigcup_{i \in I} U_i$ with U_i = open, convex $\subset \mathbb{R}^n$, I is countable.

WLOG $I = \mathbb{N}$. $\forall k \in \mathbb{N}$, put $V_k := U_1 \cup \dots \cup U_k$.

By step 2, PD is an iso for V_k, $\forall k$.

Now $M = \bigcup_{k \in \mathbb{N}} V_k$, so PD_M is an iso by claim 2.

Step 4 $M \subset \mathbb{R}^n$ is any open subset.

The topology of M has a countable basis consisting of balls. So by step 3 we are done.

Step 5 $M = \bigcup_{i \in I} U_i$ with U_i homeomorphic to open subset in $\mathbb{R}^n$ & I is countable

(We do not assume $M \subset \mathbb{R}^n$).

The proof is the same as in Steps 2, 3, 4. First prove for I-finite by induction on I and then for $I = \mathbb{N}$.

SUMMARY

If M can be covered by countably many charts, then PD_M is an iso.

Step 6 M = a general (noncompact) manifold that cannot be covered by a countable union of charts.

Use Zorn's lemma.

$T :=$ collection of all open subsets $\mathcal{U} \subset M$ s.t. $PD_{\mathcal{U}}$ is an iso

Define $\mathcal{U}' \leq \mathcal{U}''$ if $\mathcal{U}' \subset \mathcal{U}''$.

If $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is a chain in T, then $\bigcup_{\alpha \in I} \mathcal{U}_\alpha$ is also in T (by Claim 2).

So every chain in T has an upper bound. By Zorn's lemma $\exists$ a max elt. V in T. Now if $V \subsetneq M$, take a chart $\mathcal{U}$ around $x_0 \in M \setminus V$.

$PD_{\mathcal{U}}$ is an iso $\Rightarrow \mathcal{U}$ is in T.

Also $\mathcal{U} \cap V$ is in T (because $\mathcal{U} \cap V \subset \mathcal{U}$ is open). By claim 1, $\mathcal{U} \cup V$ is also in T. Contradiction to maximality of V. $\blacksquare$