Geometric idea of Poincare Duality

You saw last time what Poincare Duality is. Reminder:

<u>Thm</u> If M is a closed R-oriented topological manifold of dim n, for any coefficient ring R, there is an isomorphism

$$H^k(M;R) \underset{PD}{\cong} H_{n-k}(M;R) \qquad \forall k$$

Poincare Duality

The existence of this isomorphism is not purely due to algebra like the isomorphism in the Universal Coefficients Theorem was, but rather due to special topological properties of manifolds.

<u>Def</u> A <u>topological manifold</u> of dim n is a topological space M s.t.

1. M is Hausdorff (for any two points $\exists$ nbhds of each which are disjoint from each other)
2. M is locally Euclidean
   $\forall x \in M$ $\exists$ a nbhd $U_x \subset M$ of x & a homeo $\varphi_x : \mathbb{R}^n \approx U_x$.
3. M is second countable
   Topology has a countable base

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<u>Eg</u>
- <u>orientable</u> topological mfld : torus
- <u>non-orientable</u> topological mfld: Klein bottle, $\mathbb{R}P^n$.

<u>Def</u> A topological mfld is <u>closed</u> if it is compact and without boundary

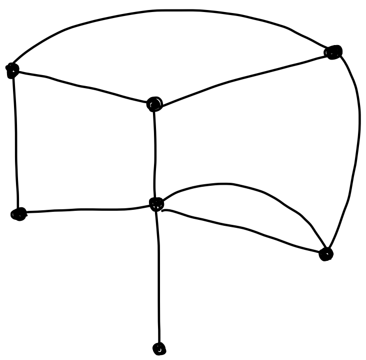

closed


not closed

<u>Motivating example</u>

Let $G \subseteq S^2$ be a graph, $G = (V, E)$

set of vertices set of edges



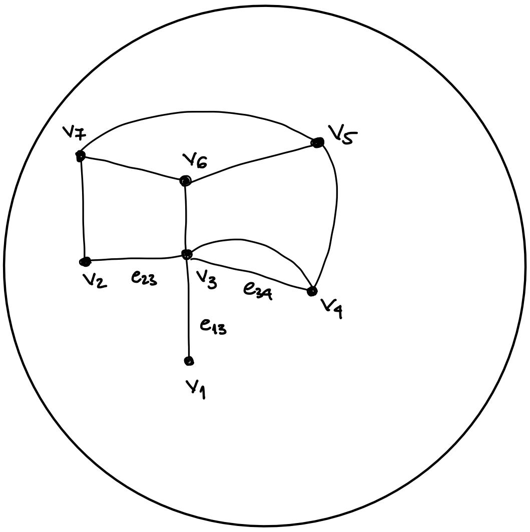
Assume G is an embedded graph ie vertices are points on S^2 and edges only intersect in common vertices.

Notation: $V = \{v_1, \dots, v_n\}$ vertices

$E = \{ e_{ij} \mid (i,j)$ is a finite multiset of pairs $i,j \in \{1, \dots, n\} \}$

ie edges can repeat

$F = \{f_1, \dots, f_\ell\}$ faces of G ie closures of components of $S^2 \setminus G$



this determines a CW decomposition of S^2.

$\#V = 7$

$\#E = 10$

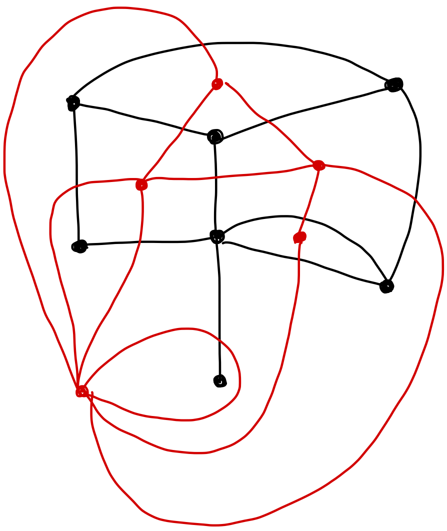
$\#F = 5$

One can associate the dual graph $\hat{G}$ to G:

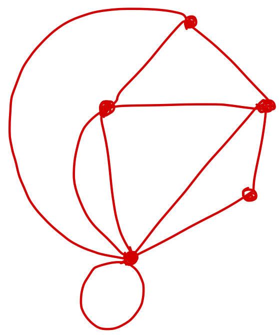
- to a 2-cell, associate a 0-cell
- — " — 1-cell — " — 1-cell
- — " — 0-cell — " — 2-cell

Generally, to an i-cell, associate an $(n-i)$-cell.

In case of G as above, we get:



Beautify



What did we do?

- a face f_k in G becomes a vertex $\hat{f}_k$ in $\hat{G}$, which is an interior point of f_k.

- an edge e_{ij} in G becomes an edge $\hat{e}_{ij}$ in $\hat{G}$:
  - $\hat{e}_{ij}$ connects vertices $\hat{f}_{k_1}$ & $\hat{f}_{k_2}$, where f_{k_1} and f_{k_2} are faces of G that have e_{ij} as their boundary;
  - e_{ij} & $\hat{e}_{ij}$ intersect once transversally.

- every vertex v_i in G belongs to exactly one face of $\hat{G}$, which we denote $\hat{v}_i$. This gives correspondence $v_i \mapsto \hat{v}_i$.

The dual graph $\hat{G}$ determines another CW structure on S^2.

Both G & $\hat{G}$ can be used to compute the homology of S^2. We want to see some connection between the chain complexes of G and of $\hat{G}$.

There exists a dualising isomorphism D from chains relative to G to <u>cochains</u> relative to $\hat{G}$:

$$\begin{array}{lccccccccc}
G: & 0 & \longrightarrow & C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\partial} & 0 \\
 & & & \downarrow D & & \downarrow D & & \downarrow D & & \\
\hat{G}: & 0 & \longrightarrow & C^0 & \xrightarrow[\delta]{} & C^1 & \xrightarrow[\delta]{} & C^2 & \xrightarrow[\delta]{} & 0
\end{array}$$

D(i-cell) = Hom-dual of the <u>dual</u> cell

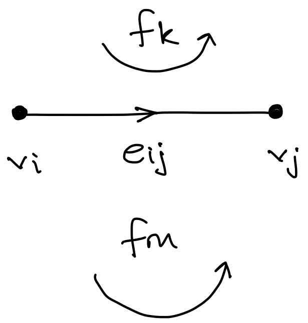
By construction, D is an isomorphism.
To show it induces an isomorphism in (co)homology, we need to verify that D is a chain map.

In other words, we need to verify the commutativity of squares:

$$D \circ \partial = \delta \circ D$$

To be able to apply δ and ∂, we need to orient the cells.

G

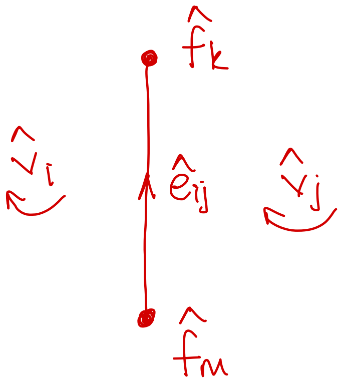


$$\left.\begin{aligned}\partial v_i &= 0\\ \partial v_j &= 0\end{aligned}\right\}\ \text{0-cells}$$

$$\partial e_{ij} = v_j - v_i \qquad \text{1-cell}$$

$$\left.\begin{aligned}\partial f_k &= e_{ij} + \dots\\ \partial f_m &= -e_{ij} + \dots\end{aligned}\right\}\ \text{2-cells}$$

$\hat{G}$



This orientation is consistent with the one on G, but we won't go into how exactly.

> Recall: for $c^* \in C^i = \mathrm{Hom}(C_i, \mathbb{Z})$, $a \in C_{i+1}$
>
> $\delta c^* \in C^{i+1} = \mathrm{Hom}(C_{i+1}, \mathbb{Z})$ $\quad \delta c^*(a) \overset{\text{def}}{=} c^*(\underbrace{\partial a}_{C_i})$

$$\left.\begin{aligned}\delta \hat{v}_i^* &= 0\\ \delta \hat{v}_j^* &= 0\end{aligned}\right\}\ \dim 2$$

(equal to 0 because there are no 3-cells)

$$\delta \hat{e}_{ij}^* = \hat{v}_j^* - \hat{v}_i^* \quad \}\ \dim 1$$

why do we have this?

$\delta \hat{e}_{ij}^* \in C^2$. If we can show that the maps $\delta \hat{e}_{ij}^*$ and $\hat{v}_j^*$ agree on elements of $C_2 = \{v_i, v_j\}$, we're done.

$$\hat{v}_j^*(\hat{v}_j) \overset{\text{def}}{=} 1$$
$$\hat{v}_i^*(\hat{v}_j) \overset{\text{def}}{=} 0$$

$$\delta \hat{e}_{ij}^*(\hat{v}_j) = \hat{e}_{ij}^*(\partial \hat{v}_j) = \hat{e}_{ij}^*(\hat{e}_{ij}) = 1$$

Similarly for $\hat{v}_i$, but with a minus:

$$\delta \hat{e}_{ij}^*(\hat{v}_i) = \hat{e}_{ij}^*(\partial \hat{v}_i) = \hat{e}_{ij}^*(-\hat{e}_{ij}) = -1$$

$$\delta \hat{f}_k^* = \hat{e}_{ij}^* + \dots$$

$$\delta \hat{f}_k^*(\hat{e}_{ij}) = \hat{f}_k^*(\partial \hat{e}_{ij}) = \hat{f}_k^*(\hat{f}_k - \hat{f}_m) = 1$$

$$\delta \hat{f}_m^* = -\hat{e}_{ij} + \dots$$

$\Rightarrow$ Indeed we get a correspondence of boundary maps $D\partial = \delta D$.

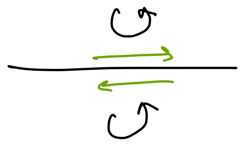
To prove this formally, we'd need to check that the orientation in the dual can be chosen consistently.

Remarks

- The construction of the dualising map D works for a graph G in any connected closed surface X (doesn't have to be orientable), as long as the faces of G are topological discs (ie G determines a regular CW decomposition of X).

- For D to be a chain map, X must be orientable.
  If we are working over $\mathbb{Z}_2$ coefficients, then this works also for any surface (all surfaces are orientable over $\mathbb{Z}_2$).

Recall: A surface X is <u>orientable</u> if orientations of 2-cells can be chosen in such a way that any two 2-cells with a common 1-cell in the boundary induce opposite orientations in this 1-cell



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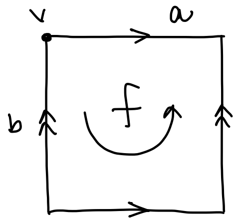
Example $\mathbb{T}^2 = S^1 \times S^1$ torus

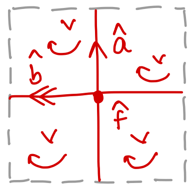
Cell structure:

$\mathbb{T}^2$ orientable ☺



$\partial v = 0$
$\partial a = v - v = 0$
$\partial b = v - v = 0$
$\partial f = a + b - a - b = 0$

Dual cell structure:



not an edge

$\delta \hat{v}^* = 0$ (no 3-cells)

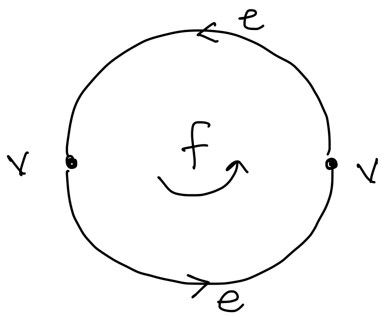
$\delta \hat{a}^* = 0 \qquad \delta \hat{a}^*(\hat{v}) = \hat{a}^*(\partial \hat{v}) = 0$
$\delta \hat{b}^* = 0 \qquad \delta \hat{b}^*(\hat{v}) = \hat{b}^*(\partial \hat{v}) = 0$

$\delta \hat{f}^* = 0 \qquad \delta \hat{f}^*(\hat{a}) = \hat{f}^*(\partial \hat{a}) = 0$
$\delta \hat{f}^*(\hat{b}) = \hat{f}^*(\partial \hat{b}) = 0$

Example: $\mathbb{R}P^2$ = real projective plane

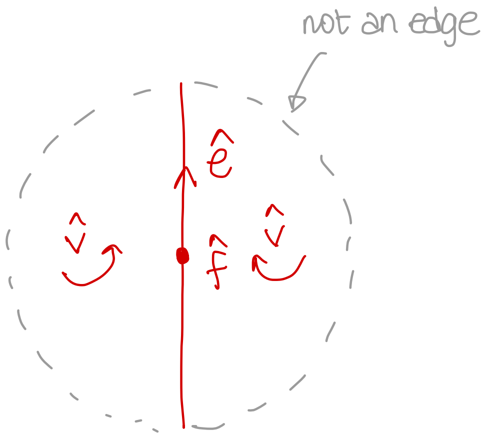
Cell structure:

$\mathbb{R}P^2$ not orientable ∴

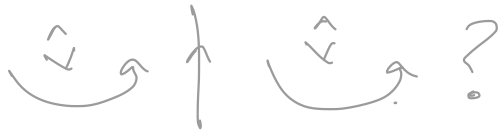


$\partial v = 0$
$\partial e = v - v = 0$
$\partial f = 2e$

Dual cell structure:



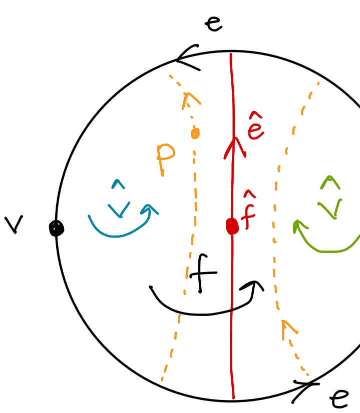
Why are $\hat{v}$'s oriented like that and not



Remember that we're in $\mathbb{R}P^2$, and that there is only one face $\hat{v}$. So we can only choose the orientation once, eg on the left.

The orientation of the "right $\hat{v}$" will follow from the geometry of $\mathbb{R}P^2$

I think about it as follows:



1. Choose the orientation of the "left" $\hat{v}$.
2. Let a point p follow the chosen orientation.
3. Deduce the orientation of the "right" $\hat{v}$.

Since $\mathbb{RP}^2$ is non-orientable, we expect to have a problem.

$$\delta\hat{v}^* = 0 \qquad (\text{no 3-cells})$$

$$\delta\hat{e}^* = 2\hat{v}^* \qquad \delta\hat{e}^*(\hat{v}) = \hat{e}^*(\partial\hat{v}) = \hat{e}^*(2\hat{e}) = 2$$

$$\delta\hat{f}^* = 0 \qquad \delta\hat{f}^*(\hat{e}) = \hat{f}^*(\partial\hat{e}) = 0$$

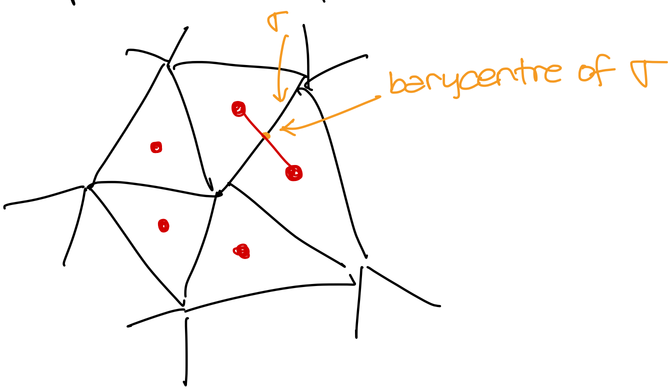
In this case we don't have the correspondence between ∂ and δ under D, no matter how we try to orient the dual cells.

Construction of dual cells in triangulated n-manifolds

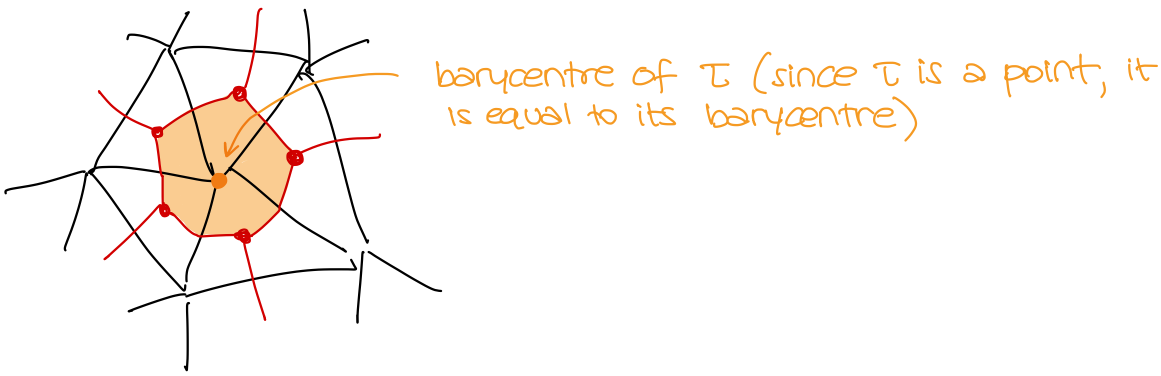
We saw examples for surfaces. But an analogous construction works in any connected oriented closed manifold that admits a triangulation by a simplicial complex K.

- dual 0-cells are barycentres of n-simplices

- dual 1-cell to $(n-1)$ simplex σ is the cone with apex the barycentre of σ and with base the 0-cells corresponding to the n-simplices whose face is σ.



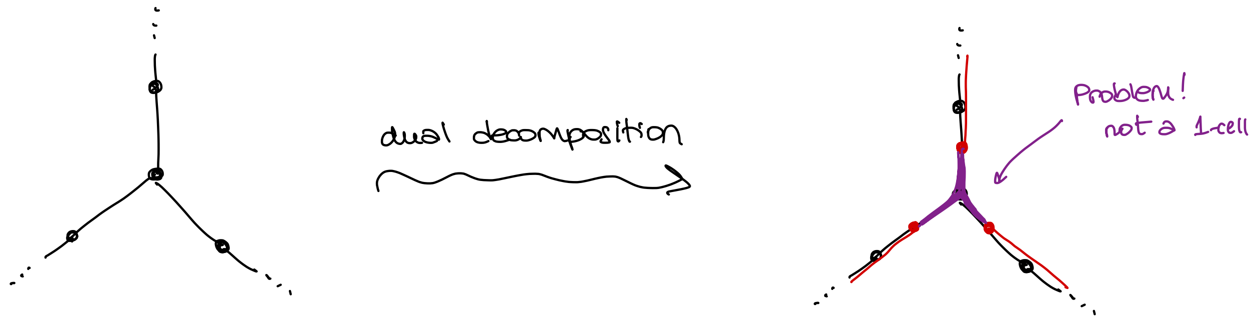
- dual 2-cell to $(n-2)$-simplex τ is the cone with apex the barycentre of τ and with base the union of 1-cells corresponding to $(n-1)$-simplices whose face is τ



(Non-) examples

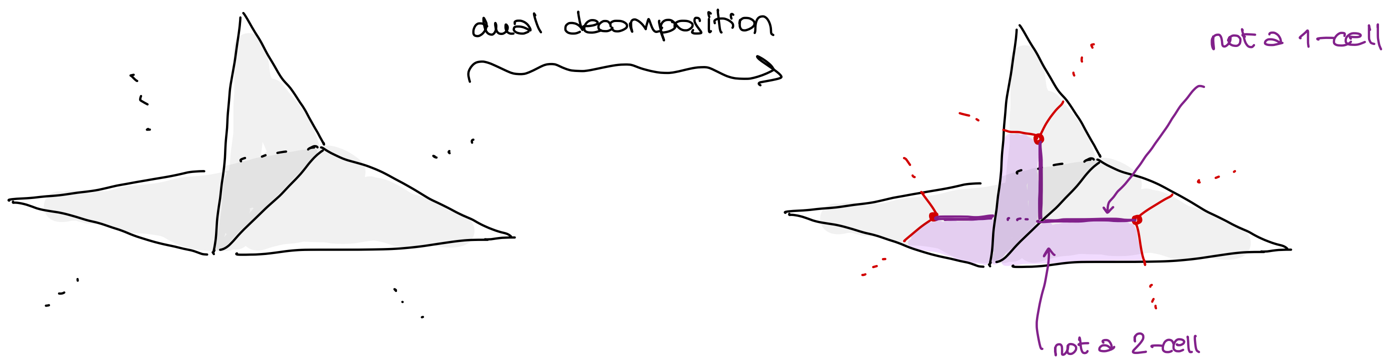
What happens if we try to find a dual of something that is not a manifold?

1)



1-dim simplicial complex

2)



2-dim simplicial complex

Topology of X imposes restrictions on the triangulation K:

1) X compact $\Rightarrow$ K finite $\Rightarrow$ $C_\bullet, C^\bullet, H_*, H^*$ finitely generated

2) X n-mfld $\Rightarrow$ principal simplexes (those that are not faces of any other simplices) are all n-dimensional
otherwise X is not locally Euclidean.

3) $\partial X = \emptyset$ $\Rightarrow$ every $(n-1)$-simplex is a face of exactly two n-simplices.

We get fundamental class with $\mathbb{Z}_2$ coefficients regardless of orientability over $\mathbb{Z}$.

4) X orientable $\Rightarrow$ n-simplexes can be oriented consistently, so that any two induce opposite orientation in a common $(n-1)$ face.

$\rightsquigarrow$ corresponding homology class ("fundamental class") is a generator of $H_n(X;\mathbb{Z})$ and denoted $[X]$

5) X n-mfld $\Rightarrow$ K has only the simplices of dimension $\leq n$
$\Rightarrow H_i(X;\mathbb{Z}) = 0$ for $i > n$.

6) X (PL) n-mfld $\Rightarrow$ dual cell of k-simplex is a topological disc of dimension $n-k$, so an $(n-k)$-cell

If we don't have the PL condition, we still end up with something which is homologically as good as discs
We can still use them for computations.

7) X orientable $\Rightarrow$ dual cells can be oriented consistently so that D is a chain map.

Exercise:

Find the dual decomposition of a (solid) tetrahedron:

Solution: