CHANGE OF COEFFICIENTS

Let $\varphi : G_1 \to G_2$ be a hom of abelian groups.

Such a φ induces a chain map

$$\varphi^c : S_\bullet(X, A; G_1) \longrightarrow S_\bullet(X, A; G_2).$$

If $f : (X, A) \to (Y, B)$ is a map, then $\exists$ a commutative square:

$$\begin{tikzcd} S_\bullet(X, A; G_1) \arrow[r, "f_c"] \arrow[d, "\varphi^c"'] \arrow[dr, phantom, "©"] & S_\bullet(Y, B; G_1) \arrow[d, "\varphi^c"] \\ S_\bullet(X, A; G_2) \arrow[r, "f_c"] & S_\bullet(Y, B; G_2) \end{tikzcd}$$

$\Rightarrow$ we get a commutative square in homology:

$$\begin{tikzcd} H_*(X, A; G_1) \arrow[r, "f_*"] \arrow[d, "\varphi_*"'] \arrow[dr, phantom, "©"] & H_*(Y, B; G_1) \arrow[d, "\varphi_*"] \\ H_*(X, A; G_2) \arrow[r, "f_*"'] & H_*(Y, B; G_2) \end{tikzcd}$$

If $0 \to G' \to G \to G'' \to 0$ is a SES of abelian groups, then we get a SES of chain complexes

$$0 \to S_\bullet(X,A;G') \to S_\bullet(X,A;G) \to S_\bullet(X,A;G'') \to 0.$$

$\Rightarrow$ We get a LES in homology:

$$\cdots \to H_n(X,A;G') \to H_n(X,A;G) \to H_n(X,A;G'')$$

$$\to H_{n-1}(X,A;G') \to \cdots$$

the connecting homomorphism is called the BOCKSTEIN HOMOMORPHISM

Two interesting examples

⊛ $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$

⊛ $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$

$$\begin{array}{ccc} \psi & & \\ k & \longmapsto & pk \end{array}$$

DEGREE THEORY WITH COEFFICIENTS IN A GROUP

Recall that if $f: S^n \to S^n$, then $\deg f = d \in \mathbb{Z}$, when $d \in \mathbb{Z}$ is the unique integer s.t.

$$\begin{array}{ccc} \tilde{H}_n(S^n) & \xrightarrow{f_*} & \tilde{H}_n(S^n) \\ \cong & & \cong \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

$$f_*(a) = d \cdot a$$

If we take coefficients in G, we get

$$\begin{array}{ccc} \tilde{H}_n(S^n; G) & \longrightarrow & \tilde{H}_n(S^n; G) \\ \cong & & \cong \\ G & & G \end{array}$$

Question: Can we still say that $f_*(a) = d \cdot a$ with $d \in \mathbb{Z}$, as before? Yes!

If we consider $\sigma_0 : \Delta^n \to \Delta^n/\partial\Delta^n \approx S^n$, then we've seen that this is a generator of $H_n(S^n, *) \cong \tilde{H}_n(S^n)$. $\Rightarrow f \circ \sigma_0$ is homologous to $d \cdot \sigma_0$. $\Rightarrow$

$$f \circ \sigma_0 - d \cdot \sigma_0 = \partial \tau \quad \text{for some}$$

$\tau : \Delta^{n+1} \to S^n$. $\Rightarrow$

$$f_c(g\sigma_0) - d(g\sigma_0) = \partial(g\tau) \text{ in } H_n(S^n, *; G)$$

$$\forall g \in G.$$

Conclusion: $f_*[g\sigma_0] = d \cdot [g\sigma_0] \ \forall g \in G$.

$\Rightarrow f_*(a) = d \cdot a \quad \forall a \in \tilde{H}_n(S^n; G)$.

With this we can now define cellular homology with coefficients in G.

CELLULAR HOMOLOGY WITH COEFFICIENTS

K CW-complex

$C_\bullet^{CW}(K)$ cellular chain complex ($\mathbb{Z}$ coefficients)

$$\to C_i^{CW}(K) \xrightarrow{d_i} C_{i-1}^{CW}(K) \xrightarrow{d_{i-1}} C_{i-2}^{CW}(K) \to \cdots$$

$$C_i^{CW}(K) = \bigoplus_{\sigma \in I_i} \mathbb{Z} \cdot \sigma$$

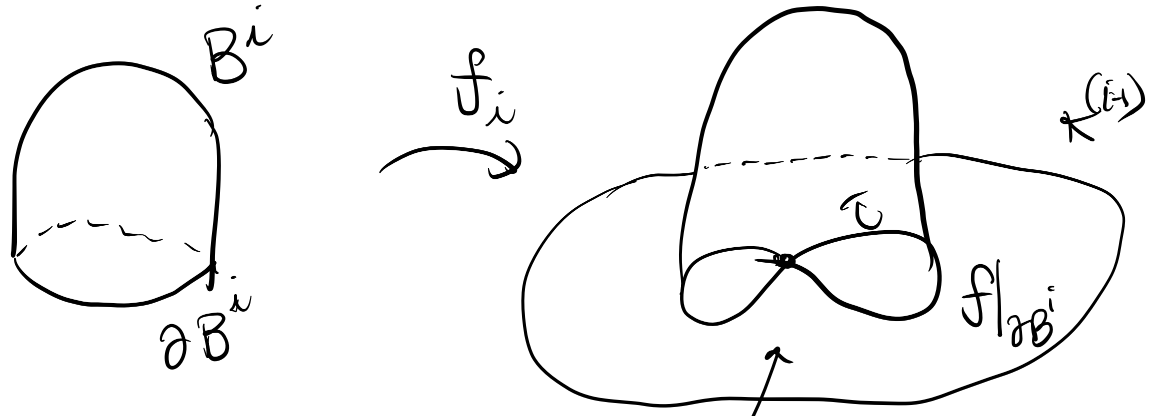
$$d_i(\sigma) = \sum_{\tau \in I_{i-1}} [\sigma : \tau]\, \tau$$

What is $[\sigma : \tau]$?

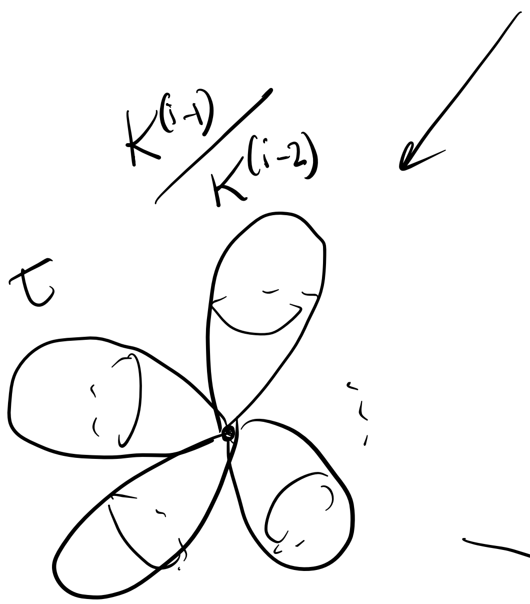
$f_i : B^i \to K$ characteristic map

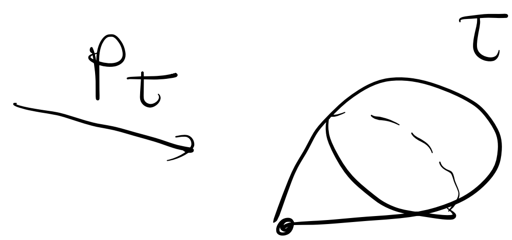
σ is attached to $K^{(i-1)}$ via $f_i|_{\partial B^i}$
attaching map

$f_i|_{\partial B^i}$ induces a map $S^{i-1} \to S^{i-1}$
as depicted below



the boundary is attached to many cells in $K^{(i-1)}$





$$[\sigma : \tau] = \deg\left(p_\tau \circ f_i|_{\partial B^i}\right).$$

To compute cellular homology with coefficients, we define the degree as above.

$$C_i^{cw}(K;G) \cong \bigoplus_{\sigma \in I_i} G \cdot \sigma$$

EXAMPLE

Consider $\mathbb{R}P^n = S^n \big/ \underset{\forall x \in S^n}{x \sim -x}$.

$\mathbb{R}P^n$ has CW structure with one cell in each dim $0 \le i \le n$. The $(i-1)$-skeleton of $\mathbb{R}P^i$ is $\mathbb{R}P^{i-1}$, the i-th skeleton, $\mathbb{R}P^i$, is obtained $\mathbb{R}P^i = \mathbb{R}P^{i-1} \cup_{\partial} B^i$ with attaching map $f_i : \underset{S^{i-1}}{\underset{\shortparallel}{\partial B^i}} \to \mathbb{R}P^{i-1}$

by $f_i(x) = [x]$.

$([x] = [-x])$

The cellular chain complex $C_\bullet^{CW}(\mathbb{R}P^n)$ has $C_i^{CW} = \mathbb{Z} \cdot e^{(i)}$ $\forall\ 0 \le i \le n$.

$\uparrow$ generator

$\partial : C_i^{CW} \to C_{i-1}^{CW}$ is $\partial(e^{(i)}) = q \cdot e^{(i-1)}$,

where

$$q = \begin{cases} 0 & i = \text{odd} \\ 2 & i = \text{even} \ge 2 \\ 0 & i \le 0,\ i > n \end{cases}$$

reminder: $\mathbb{R}P^{i-1} / \mathbb{R}P^{i-2} \approx S^{i-1}$

So where does g come from?
Each f_i has two preimages:
signs depend on the degree
of the antipodal map: $(-1)^i$ for
$S^{i-1} \to S^{i-1}$ $\Rightarrow$ the degree of g is
$1 + (-1)^i$ (the projection map is
just the identity since $\mathbb{R}P^{i-1} / \mathbb{R}P^{i-2} \approx S^{i-1}$)

From here we get:

(*) If n = even

$$H_i(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} 0 & i > n \\ 0 & 2 \leq i = \text{even} \leq n \\ \mathbb{Z}_2 & 1 \leq i = \text{odd} < n \\ \mathbb{Z} & i = 0 \end{cases}$$

(*) If $n = \text{odd}$

$$H_i(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} 0 & i > n \\ \mathbb{Z} & i = n \\ 0 & 2 \leq i = \text{even} \leq n \\ \mathbb{Z}_2 & 1 \leq i = \text{odd} < n \\ \mathbb{Z} & i = 0 \end{cases}$$

What happen for other G's?

$C_i^{cw}(\mathbb{R}P^n; G) = G \cdot e^{(i)}$

$\forall\, 0 \leq i \leq n$, $C_i^{cw} = 0$ for $i > n$

& $i \leq 0$

$\partial : C_i^{cw} \to C_{i-1}^{cw}$ is multiplication by

g as before. $\Rightarrow$

$$d : C_i^{cw}(\mathbb{R}P^n; G) \to C_{i-1}^{cw}(\mathbb{R}P^n; G)$$

If $0 < i = \text{even} \leq n$ $\quad d(a \cdot e^{(i)}) = 2a\, e^{(i-1)}$ $\quad \forall a \in G$.

If $0 < i = \text{odd} \leq n \Rightarrow d(a e^{(i)}) = 0$

$$Z_i = \begin{cases} K\cdot e^{(i)} & 0<i=\text{even}\le n \\ G\cdot e^{(i)} & 0<i=\text{odd}\le n \\ G\cdot e^{(0)} & i=0 \\ 0 & i<0 \text{ or } i>n \end{cases}$$

cycles $\subset C_i$

where $K := \ker\left(G \xrightarrow{2x} G\right) \subset G$

$$B_i = \begin{cases} 0 & i=n \\ 0 & 0<i=\text{even}\le n \\ 2G\cdot e^{(i)} & 0<i=\text{odd}<n \\ 0 & i=0 \\ 0 & i<0 \text{ or } i>n \end{cases}$$

$\parallel$

$d(C_{i+1})$

boundaries

Now we can calculate homology groups

$$H_i^{cw}(\mathbb{R}P^n; G) \cong \begin{cases} K & 0<i=\text{even}<n \\ G/2G & 0<i=\text{odd}<n \\ G & i=0 \\ 0 & i<0 \text{ or } i>n \end{cases}$$