where $K := \mathrm{Ker}(G \xrightarrow{2x} G) \subset G$

$$B_i = \begin{cases} 0 & i = n \\ 0 & 0 < i = \text{even} \leq n \\ 2G \cdot e^{(i)} & 0 < i = \text{odd} < n \\ 0 & i = 0 \\ 0 & i < 0 \text{ or } i > n \end{cases}$$

$\parallel$

$d(C_{i+1})$

boundaries

Now we can calculate homology groups

$$H_i^{cw}(\mathbb{R}P^n; G) \cong \begin{cases} K & 0 < i = \text{even} < n \\ G/2G & 0 < i = \text{odd} < n \\ G & i = 0 \\ 0 & i < 0 \text{ or } i > n \end{cases}$$

$$H_n^{cw}(\mathbb{R}P^n; G) \cong \begin{cases} K & n = \text{even} \\ G & n = \text{odd} \end{cases}$$

Several interesting examples of G

① $G = \mathbb{Z}$ $\qquad 2G = 2\mathbb{Z} \subset \mathbb{Z}$

$K = 0$

Conclusion:

$$H_i^{cw}(\mathbb{R}P^n; G) \cong \begin{cases} 0 & 0 < i = \text{even} < n \\ \mathbb{Z}_2 & 0 < i = \text{odd} < n \\ \mathbb{Z} & i = 0 \\ 0 & i < 0 \text{ or } i > n \end{cases}$$

$$H_n(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} 0 & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases}$$

② Assume $\forall g \in G$, $\exists ! h \in G$ s.t.

unique h

$$2h = g$$

(e.g. $G = \mathbb{Q}$, $G = \mathbb{R}$, $G = \mathbb{C}$ or any field of char $\neq 2$)

Then $G \xrightarrow{2\times} G$ is an isomorphism.

In this case $K=0$, $2G=G$, so $G/2G=0$.

$H_0(\mathbb{R}P^n;G)\cong G$, $H_i(\mathbb{R}P^n;G)=0$ $0<i<n$

$$H_n(\mathbb{R}P^n;G)=\begin{cases} 0 & n=\text{even} \\ G & n=\text{odd} \end{cases}$$

③ $G=\mathbb{Z}_2$

In this case $2G=0$, $K=\mathbb{Z}_2$, $G/2G=\mathbb{Z}_2$

$$H_i(\mathbb{R}P^n;\mathbb{Z}_2)\cong\mathbb{Z}_2 \quad \forall\ 0\le i\le n$$

APPLICATION: BORSUK-ULAM THEOREM

THEOREM vector-valued function

Let $f:S^n\to\mathbb{R}^n$ be a continuous map $\Rightarrow$

$\exists\ x\in S^n$ s.t. $f(x)=f(-x)$.

EXAMPLE

Take $n=2$, $S^2=$ surface of Earth at fixed time t_0

$$f(x)=\left(\text{temp}_{t_0}(x),\ \text{press}_{t_0}(x)\right)$$

PROOF

IMPORTANT to keep in mind:

$$H_i(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \forall\, 0 \leq i \leq n.$$

Let $\pi: X \to Y$ be a $2:1$ covering.

Let $\theta: X \to X$ be the unique deck transformation s.t. $\theta \neq id$. So $\theta(x) \neq x \ \forall x \in X$.

$$\theta \circ \theta = id.$$

Example. $X = S^n$, $Y = \mathbb{R}P^n = S^n/_{x \sim -x}$ $\quad \theta(x) = -x$.

We'll work now with $S_\bullet(X; \mathbb{Z}_2)$ and $S_\bullet(Y; \mathbb{Z}_2)$.

Let $\lambda: \Delta^k \to X$ be a k-simplex.

$\Rightarrow \theta \circ \lambda$ is a different simplex

Let $\sigma: \Delta^k \to Y$ be a k-simplex in Y.

σ can be lifted to $\tilde{\sigma}: \Delta^k \to X$.

$\exists$ exactly two possible such liftings:

$$\tilde{\sigma} \text{ and } \theta \circ \tilde{\sigma}.$$

($\exists$ a lifting since Δ^k is simply connected)

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ \Delta^k \arrow[ur, dashed, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

Define $T : S_\bullet(Y;\mathbb{Z}_2) \longrightarrow S_\bullet(X;\mathbb{Z}_2)$

$$\sigma \overset{T}{\longmapsto} \tilde{\sigma} + \theta\circ\tilde{\sigma}$$

(this is independent of the choice of the lift $\tilde{\sigma}$)

CLAIM

T is a chain map.

Proof

Exercise.

CLAIM

T fits into the following SES of chain complexes

$$0 \to S_\bullet(Y;\mathbb{Z}_2) \xrightarrow{T} S_\bullet(X;\mathbb{Z}_2) \xrightarrow{\pi_c} S_\bullet(Y;\mathbb{Z}_2) \to 0.$$

For the exactness it is crucial to work with $\mathbb{Z}_2$-coefficients ($\pi_c \circ T(\sigma) = 2\sigma$).

Proof

Exercise.

This SES induces a LES in homology:

$$\cdots \to H_k(Y;\mathbb{Z}_2) \xrightarrow{T_*} H_k(X;\mathbb{Z}_2) \xrightarrow{\pi_*} H_k(Y;\mathbb{Z}_2) \xrightarrow{\partial_*} H_{k-1}(Y;\mathbb{Z}_2) \to$$

(SMITH EXACT SEQUENCE
GYSIN SEQUENCE)

Suppose we have two coverings, each of them 2:1.

$$X \xrightarrow{\pi} Y$$
$$X' \xrightarrow{\pi'} Y'$$

and we have the deck transformations

$\theta : X \to X$, $\theta' : X' \to X'$.

Let $f : X \to X'$ be a map s.t. $f \circ \theta = \theta' \circ f$

(f is an odd map in the previous example)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X' \arrow[d, "\pi'"] \\ Y \arrow[r, "\bar{f}"] & Y' \end{tikzcd}$$

f descends to $\bar{f} : Y \to Y'$.

We get a map of SESs induced by f & $\bar{f}$:

$$\begin{tikzcd} 0 \arrow[r] & S_\bullet(Y;\mathbb{Z}_2) \arrow[r, "\tau"] \arrow[d, "\bar{f}_c"] & S_\bullet(X;\mathbb{Z}_2) \arrow[r, "\pi_c"] \arrow[d, "f_c"] & S_\bullet(Y;\mathbb{Z}_2) \arrow[r] \arrow[d, "\bar{f}_c"] & 0 \\ 0 \arrow[r] & S_\bullet(Y';\mathbb{Z}_2) \arrow[r, "\tau'"] & S_\bullet(X';\mathbb{Z}_2) \arrow[r, "\pi'_c"] & S_\bullet(Y';\mathbb{Z}_2) \arrow[r] & 0 \end{tikzcd}$$

Exercise: Check the commutativity of this diagram.

Take $X = S^n$, $Y = \mathbb{R}P^n$, $X' = S^m$, $Y' = \mathbb{R}P^m$,

θ, θ' are antipodal maps.

THEOREM

Let $\phi : S^n \to S^m$ be an odd map

(i.e. $\phi(-x) = -\phi(x)$, or equiv. $\phi \circ \theta = \theta' \circ \phi$).

Then $n \leq m$.

PROOF

Assume by contradiction that $n > m$.

WLOG assume that $m > 0$, bc. if $m = 0$, the statement is obvious: $\nexists$ odd map $S^n \to S^0$ if $n > 0$.

Consider

$$\begin{tikzcd} S^n \arrow[r, "\phi"] \arrow[d, "\pi"'] & S^m \arrow[d, "\pi'"] \\ \mathbb{R}P^n \arrow[r, "\bar{\phi}"'] & \mathbb{R}P^m \end{tikzcd}$$

Consider the LES, discussed before, for $S^m \to \mathbb{R}P^m$:

$$0 \to H_m(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\tau_*'} H_m(S^m;\mathbb{Z}_2) \xrightarrow{\pi_*'} H_m(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\partial_*}$$

$$\to H_{m-1}(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\tau_*'} H_{m-1}(S^m;\mathbb{Z}_2) \xrightarrow{\pi_*'} H_{m-1}(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\partial_*}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\to H_1(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\tau_*'} H_1(S^m;\mathbb{Z}_2) \xrightarrow{\pi_*'} H_1(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\partial_*}$$

$$\to H_0(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\tau_*'} H_0(S^m;\mathbb{Z}_2) \xrightarrow{\pi_*'} H_0(\mathbb{R}P^m;\mathbb{Z}_2) \to 0$$

CLAIM

$\partial_* : H_k(\mathbb{R}P^m;\mathbb{Z}_2) \to H_{k-1}(\mathbb{R}P^m;\mathbb{Z}_2)$

is an iso $\forall\ 1 \le k \le m$.

injective $\mathbb{Z}_2 \to \mathbb{Z}_2$ map is iso

$$0 \to \overset{\mathbb{Z}_2}{H_m(\mathbb{R}P^m;\mathbb{Z}_2)} \underset{\cong}{\xrightarrow{\tau_*'}} \overset{\mathbb{Z}_2}{H_m(S^m;\mathbb{Z}_2)} \xrightarrow{\pi_*'} H_m(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\partial_*}$$

$$\to \overset{\mathbb{Z}_2}{H_{m-1}(\mathbb{R}P^m;\mathbb{Z}_2)} \xrightarrow{\tau_*'} \overset{0}{H_{m-1}(S^m;\mathbb{Z}_2)} \xrightarrow{\pi_*'} H_{m-1}(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\partial_*}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\to \overset{\mathbb{Z}_2}{H_1(\mathbb{R}P^m;\mathbb{Z}_2)} \xrightarrow{\tau_*'} \overset{0}{H_1(S^m;\mathbb{Z}_2)} \xrightarrow{\pi_*'} H_1(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\partial_*}$$

$$\to \overset{\mathbb{Z}_2}{H_0(\mathbb{R}P^m;\mathbb{Z}_2)} \xrightarrow{\tau_*'} \overset{\mathbb{Z}_2}{H_0(S^m;\mathbb{Z}_2)} \xrightarrow{\pi_*'} H_0(\mathbb{R}P^m;\mathbb{Z}_2) \to 0$$

Exercise: write down the proof carefully.

The same happens for the sequence $S^n \to \mathbb{R}P^n$ (this time the range is $0 \leq k \leq n$).

Now we look at the relationship between the sequences:

$$\begin{tikzcd} H_i(\mathbb{R}P^n; \mathbb{Z}_2) \arrow[r, "\partial_*"] \arrow[d, "\Phi_*"'] & H_{i-1}(\mathbb{R}P^n; \mathbb{Z}_2) \arrow[d, "\Phi_*"] \\ H_i(\mathbb{R}P^m; \mathbb{Z}_2) \arrow[r, "\partial_*"] & H_{i-1}(\mathbb{R}P^m; \mathbb{Z}_2) \end{tikzcd}$$

Begin with $i=1$: Φ_* on RHS is an iso.

Because ∂_* are isos we get that Φ_* on LHS is also an iso. Applying this argument repeatedly we get that

$$\Phi_*: H_i(\mathbb{R}P^n; \mathbb{Z}_2) \to H_i(\mathbb{R}P^m, \mathbb{Z}_2)$$

for all $0 \leq i \leq m$.

In particular,

$$\Phi_* : H_m(\mathbb{R}P^n; \mathbb{Z}_2) \to H_m(\mathbb{R}P^m; \mathbb{Z}_2)$$

is an iso.

$\Rightarrow$

$$\begin{tikzcd}
H_m(\mathbb{R}P^n; \mathbb{Z}_2) \arrow[r, "T_*"] \arrow[d, "\Phi_*\ \cong"] & H_m(S^n; \mathbb{Z}_2) = 0 \arrow[d, "\phi_*"] \\
H_m(\mathbb{R}P^m; \mathbb{Z}_2) \arrow[r, "\cong", "T_*'"'] & H_m(S^m; \mathbb{Z}_2) = \mathbb{Z}_2
\end{tikzcd}$$

LES from before

This is a contradiction.

■

PROOF OF THE BORSUK-ULAM THEOREM

Let $f: S^n \to \mathbb{R}^n$. Assume by contradiction that $f(x) \neq f(-x)$ $\forall x \in S^n$. Define

$\phi : S^n \to S^{n-1}$