In particular,

$$\Phi_* : H_m(\mathbb{RP}^n;\mathbb{Z}_2) \to H_m(\mathbb{RP}^m;\mathbb{Z}_2)$$

is an iso.

$\Rightarrow$

$$\begin{tikzcd}
H_m(\mathbb{RP}^n;\mathbb{Z}_2) \arrow[r, "T_*"] \arrow[d, "\Phi_*\ \cong"] & H_m(S^n;\mathbb{Z}_2) = 0 \arrow[d, "\phi_*"] \\
H_m(\mathbb{RP}^m;\mathbb{Z}_2) \arrow[r, "\cong", "T'_*"'] & H_m(S^m;\mathbb{Z}_2) = \mathbb{Z}_2
\end{tikzcd}$$

LES from before

This is a contradiction.

$\blacksquare$

PROOF OF THE BORSUK-ULAM THEOREM

Let $f : S^n \to \mathbb{R}^n$. Assume by contradiction that $f(x) \neq f(-x)$ $\forall x \in S^n$. Define

$\phi : S^n \to S^{n-1}$

$$\phi(x) := \frac{f(x)-f(-x)}{|f(x)-f(-x)|} \in S^{n-1}$$

Clearly, $\phi(-x) = -\phi(x)$. By the previous theorem, $n \leq n-1$. Contradiction.

$\blacksquare$

APPLICATION OF BORSUK-ULAM THEOREM:

THEOREM (LUSTERNIK-SCHNIRELMANN)

Let $A_1, \dots, A_\ell \subset S^n$ be ℓ closed subsets s.t. $A_1 \cup A_2 \cup \dots \cup A_\ell = S^n$. If $\ell \leq n+1$, then $\exists i$ s.t. A_i contains a pair of antipodal points.

Proof

WLOG $\ell = n+1$ (otherwise add empty subsets: $A_{\ell+1}, \dots, A_{n+1} = \emptyset$).

Assume $A_i \cap (-A_i) = \emptyset \quad \forall 1 \leq i \leq n$

image under the

antipodal map

and we'll show that $A_{n+1} \cap (-A_{n+1}) \neq \emptyset$.

Reminder:
If X T_2 + compact,
then X is normal.
(S^n is normal)

We will need:

URYSOHN LEMMA

Let X be a normal space and $C \subset X$ closed, $\mathcal{U}$ an open subset containing C. Then $\exists$ a continuous function $f: X \to [0,1]$ s.t. $f|_C \equiv 0$ and $f|_{X \setminus \mathcal{U}} \equiv 1$.

By the Urysohn lemma $\exists$ a continuous function $f_i: X \to [0,1]$ s.t. $f|_{A_i} \equiv 0$ and $f|_{-A_i} \equiv 1$. Take the functions $f_1, f_2, \ldots, f_n$ and define $f: S^n \to \mathbb{R}^n$,

$$f(x) = (f_1(x), \ldots, f_n(x)).$$

By Borsuk-Ulam $\exists\, x_0 \in S^n$ s.t.

$$f(x_0) = f(-x_0).$$

Clearly, $x_0 \notin A_i \quad \forall\, 1 \le i \le n$. Similarly $-x_0 \notin A_i \quad \forall 1 \le i \le n$. Since A_i are a covering, $x_0, -x_0 \in S^n \setminus (A_1 \cup .. \cup A_n) \subset A_{n+1}$

$\blacksquare$

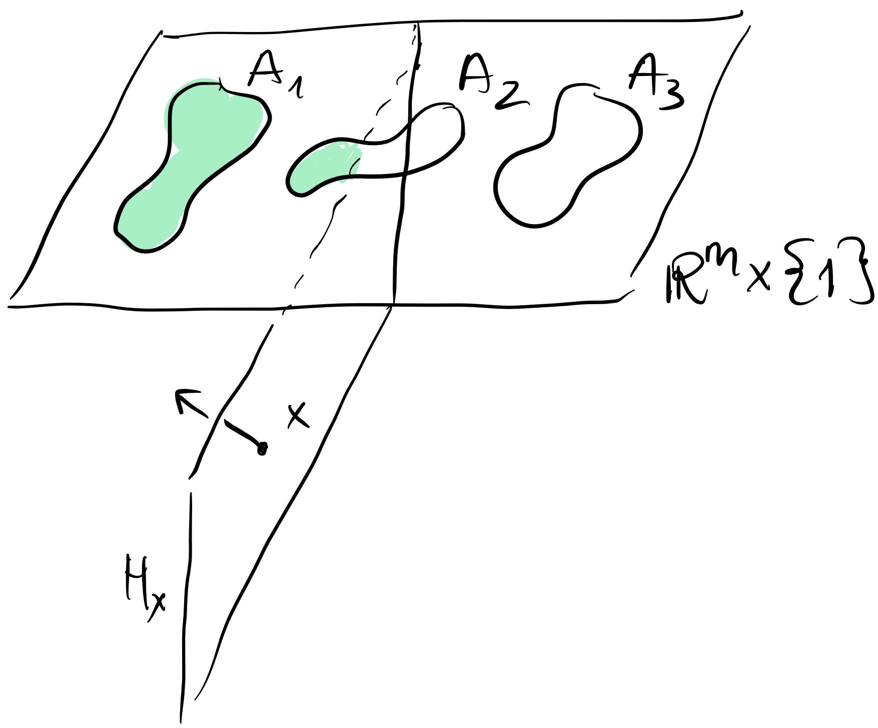
A ham sandwich consists of two pieces of bread & one piece of ham. the following theorem says that one can always slice it with a straight cut of a knife so as to cut each slice of bread exactly in two and the same for the ham.

APPLICATION: THE HAM SANDWICH THEOREM

Let $A_1, A_2, .., A_m$ be Lebesgue measurable bounded subsets of $\mathbb{R}^m$.

then there exists an affine $(m-1)$-plane $H \subset \mathbb{R}^m$ which divides each A_i into pieces of equal measure.

Proof Regard $\mathbb{R}^m$ as $\mathbb{R}^m \times \{1\} \subset \mathbb{R}^{m+1}$;



i.e. the subset $\{(x_1, \dots, x_{m+1}) \mid x_{m+1} = 1\}$

For a unit vector $x \in \mathbb{R}^{m+1}$, let

$$V_x = \mathbb{R}^n \times \{1\} \cap \{y \in \mathbb{R}^{m+1} \mid \langle x, y \rangle \geq 0\}$$

and

$$H_x = \mathbb{R}^m \times \{1\} \cap \{y \in \mathbb{R}^{m+1} \mid \langle x, y \rangle = 0\}$$

Let $f_i = \text{measure}(V_x \cap A_i)$ which is continuous since A_i is bounded.

Then put $f = (f_1, \dots, f_m) : S^m \to \mathbb{R}^m$.

By the Borsuk-Ulam theorem, there is such a vector x_0 s.t. $f(x_0) = f(-x_0)$.

Then H_{x_0} is the desired hyperplane.

$\blacksquare$