COHOMOLOGY

Let's start with algebra!

AT 1: chain complexes & homology

Now we consider cochain complexes:

DEFINITION

A **COCHAIN COMPLEX** is a sequence of abelian groups & homomorphisms:

$$\cdots \longrightarrow C^{i-1} \xrightarrow{\delta} C^{i} \xrightarrow{\delta} C^{i+1} \longrightarrow \cdots \qquad \delta \circ \delta = 0.$$

i-th COHOMOLOGY GROUP is

$$H^i(C^\bullet) := \frac{\overbrace{\ker(C^i \xrightarrow{\delta} C^{i+1})}^{\text{cocycles}}}{\underbrace{\operatorname{Im}(C^{i-1} \xrightarrow{\delta} C^i)}_{\text{coboundaries}}}$$

COCHAIN MAPS

$f : C^\bullet \longrightarrow D^\bullet$ are such maps that

$$f \circ \delta_C = \delta_D \circ f.$$

A cochain map f induces a map between cohomology groups.

* SES of cochain complexes gives rise to LES in cohomology:

$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ SES of chain complexes $\rightsquigarrow$ a LES in cohomology

$$\cdots \to H^i(A^\bullet) \xrightarrow{f^*} H^i(B^\bullet) \xrightarrow{g^*} H^i(C^\bullet) \xrightarrow{\delta^*} H^{i+1}(A^\bullet) \to \cdots$$

* COCHAIN HOMOTOPY

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i-1} \arrow[r] & C^i \arrow[r] \arrow[dl, "h"'] & C^{i+1} \arrow[r] \arrow[dl, "h"'] & \cdots \\
\cdots \arrow[r] & D^{i-1} \arrow[r] & D^i \arrow[r] & D^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

We say that cochain maps $f, g : C^\bullet \to D^\bullet$ are homotopic if $\exists\, h : C^\bullet \to D^{\bullet-1}$ s.t.

$$h \circ \delta_C + \delta_D \circ h = f - g.$$

If f & g are cochain homotopic $\Rightarrow$ $f^* = g^* : H^*(C^\bullet) \to H^*(D^\bullet)$.

Remarks: If $(C_\bullet, d)$ is a chain complex, then $(D^i : C_{-i}, \delta = d)$ is a cochain complex. It also works vice versa.

Another method

Let $(C_\bullet, \partial)$ be a chain complex, and let's fix an abelian group G.

$$D^k := \hom(C_k, G).$$

Define $\delta : D^k \to D^{k+1}$, $\delta := \partial^*$

$$\delta(f) = f \circ \partial \quad \forall f \in \hom(C_k, G).$$

$$\begin{tikzcd} \arrow[r] & C_{k+1} \arrow[r, "\partial"] \arrow[dr, "\delta(f)"'] & C_k \arrow[r] \arrow[d, "f"] & \cdots \\ & & & \end{tikzcd}$$

We have $\delta^2(f) = \delta(f \circ \partial) = f \circ \partial \circ \partial = 0$.

We get a cochain complex

$$(D^\bullet, \delta) \rightsquigarrow H^*(D^\bullet, \delta).$$

REMARK

① $f: C_k \to G$ is a cocycle $\iff f|_{\partial(C_{k+1})} \equiv 0$, ie.

$$f|_{B_k} \equiv 0.$$

② If f = coboundary, ie. $f = \delta(g)$. (i.e. $f = g \circ \partial$ for some g) $\Rightarrow f|_{Z_k} \equiv 0$.

A bit about cohomology of topological spaces.

Let X be a space, $A \subset X$ a subspace, G abelian group.

$$S^k(X;G) := \hom(S_k(X), G)$$

$$S^k(X,A;G) := \hom(S_k(X,A), G)$$

singular chains over $\mathbb{Z}$

Take $\delta = \partial^*$, as before. Homology of these chain complexes is $H^*(X;G)$ & $H^*(X,A;G)$.

SINGULAR COHOMOLOGY GROUPS

A cochain $\varphi \in S^k(X;G)$ assigns an element in G to every simplex

$\sigma : \Delta^k \to X$: $\varphi(\sigma) \in G$, or $\langle \varphi, \sigma \rangle \in G$.

$$\langle \delta\varphi, \tau \rangle := \langle \varphi, \partial\tau \rangle = \sum_{i=0}^{k+1} (-1)^i \langle \varphi, \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{k+1}]} \rangle$$

$\tau : \Delta^{k+1} \to X$

EXAMPLES

$H^0(X; G)$

function

$\varphi \in S^0(X; G)$

$\varphi : X \to G$

$$S^0(X;G) \xrightarrow{\delta} S^1(X;G)$$

$$\langle \delta\varphi, \sigma \rangle = \langle \varphi, \partial\sigma \rangle = \langle \varphi, \sigma(v_1) - \sigma(v_0) \rangle$$

a singular simplex

$\Delta^1 \to X$

$\sigma : [v_0, v_1] \to X$

$$= \varphi(\sigma(v_1)) - \varphi(\sigma(v_0))$$

$\delta\varphi = 0 \iff \varphi$ is constant on each path-connected component of X

$$\Rightarrow H^0(X;G) \cong \prod_{C \in \pi_0(X)} G$$

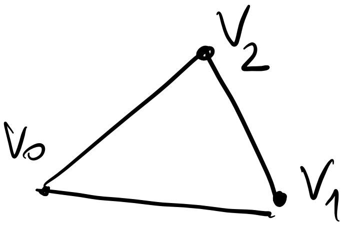
(Recall: $H_0(X;G) = \bigoplus_{C \in \pi_0(X)} G$)

Cocycles in degree 1

$\varphi \in S^1(X;G)$

$\varphi : S_1(X) \to G$, so φ assigns an element in G to every path $\tau : [0,1] \to X$.

Let $\sigma : \Delta^2 \to X$,



$$\langle \delta\varphi, \sigma \rangle = \langle \varphi, \partial\sigma \rangle =$$

$$= \langle \varphi, \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]} \rangle$$

So $\delta\varphi = 0 \iff$

$$\varphi(\sigma|_{[v_0,v_1]}) + \varphi(\sigma|_{[v_1,v_2]}) = \varphi(\sigma|_{[v_0,v_2]})$$

Let $C_\bullet$ & $D_\bullet$ be chain complexes, $\varphi : C_\bullet \to D_\bullet$ a chain map, G abelian group. Applying hom induces a cochain map

$$\varphi^* : \hom(D_\bullet, G) \to \hom(C_\bullet, G).$$

$$\varphi^*(\alpha) = \alpha \circ \varphi \quad \forall \alpha \in \hom(D_\bullet, G)$$

$\Rightarrow \varphi^*$ induces a map in cohomology.

IMPORTANT EXAMPLE

Let $f : X \to Y$ be a map between spaces.

f induces $f_c : S_\bullet(X) \to S_\bullet(Y)$, a chain map

Applying hom yields $f_c^* : S^\bullet(Y;G) \to S^\bullet(X;G)$, a cochain map. This f_c^* induces

$$f^* : H^*(Y;G) \to H^*(X;G).$$

Special case: $A \subset X$ subspace, $i : A \to X$ inclusion. What is i_c^*? If $\alpha : S_k(X) \to G$, then $i_c^*(\alpha)$ is just $\alpha|_{S_k(A)} : S_k(A) \longrightarrow G$.

THE UNIVERSAL COEFFICIENTS THEOREM

Recall a topic from last semester:

SPLIT EXACT SEQUENCES

Let R be a commutative ring (with a unit)

DEFINITION

Let $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be a SES of R-modules. We say that the sequence splits if $\exists$ iso $k : V \xrightarrow{\cong} U \oplus W$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, equal] & V \arrow[r, "g"] \arrow[d, "k", "\cong"'] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \arrow[r, "i"] & U \oplus W \arrow[r, "p"] & W \arrow[r] & 0
\end{tikzcd}$$

$i(u) := (u, 0)$, $p(u, w) := w$.

PROPOSITION

A SES $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ splits iff one of the following holds:

① $\exists$ a homomorphism $V \xleftarrow{s} W$ s.t. $g \circ s = id_W$.

② $\exists$ a homomorphism $U \xleftarrow{\pi} V$ s.t. $\pi \circ f = id_U$.

PROOF

Exercise.

EXAMPLE

$R = \mathbb{Z}$. The sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ does <u>NOT</u> split.

PROPOSITION

Let W be a free R-module. Then $\forall$ SES $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ splits

Exercise.

EXACTNESS & HOM

Let M be an R-module. Consider $\hom_R(-,M)$.

Question: Does $\hom_R(-,M)$ preserve exactness of SESs?

We will abbreviate $\hom_R$ to $\hom$.

PROPOSITION

If $U \xrightarrow{f} V \xrightarrow{g} W \to 0$ is an exact sequence, then

$$\hom(U,M) \xleftarrow{f^*} \hom(V,M) \xleftarrow{g^*} \hom(W,M) \leftarrow 0$$

is also exact.

<u>But</u>

If $0 \to U \xrightarrow{f} V$ is exact (ie. $f: U \to V$ is injective), then $0 \leftarrow \hom(U,M) \xleftarrow{f^*} \hom(V,M)$ might <u>NOT</u> be exact (f^* might not be surjective).

For example, $R=\mathbb{Z}$

$$0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \quad (m \neq 1,-1,0) \Rightarrow$$

$$\hom(\mathbb{Z},M) \leftarrow \hom(\mathbb{Z},M)$$

$$m\cdot 6 \longleftarrow\!\shortmid\ \overset{\psi}{6}$$

$M=\mathbb{Z}$, then this is multiplication by m, which is not surjective.

CONCLUSION

$\hom(-,M)$ does <u>NOT</u> preserve SES.

PROPOSITION

If $0\to U\to V\to W\to 0$ is a split SES, then $\forall$ R-mod M

$$0\leftarrow \hom(U,M)\leftarrow \hom(V,M)\leftarrow \hom(W,M)\leftarrow 0$$

is exact and moreover this sequence is also split.

Outline of PROOF WLOG we may

assume that our original sequence is $0 \to U \xrightarrow{i} U \oplus W \xrightarrow{p} W \to 0$ with $i(u) = (u, 0)$, $p(u, w) = w$.

Exercise: Justify this step.

$$0 \leftarrow \hom(U, M) \leftarrow \hom(U \oplus V, M) \leftarrow \hom(W, M) \leftarrow 0$$

$$\hom(U \oplus V, M) = \hom(U,M) \oplus \hom(V,M) \xrightarrow{g} \hom(U,M)$$

$$g(\sigma, \tau) = \sigma$$

we have exactness here since g is surjective ∎

From now on, $R = \mathbb{Z}$, so we work with abelian groups. Let $(C_\bullet, \partial)$ be a chain complex of free abelian groups.

Fix an abelian group G & consider the cochain complex $(C^*, \delta) = (\hom(C_\bullet, G), \partial^*)$.

Denote the cohomology of the latter by $H^*(C;G)$. What is the relation between $H^*(C;G)$ & $H_*(C_\cdot)$?

CLAIM

There exists an obvious map

$h: H^n(C;G) \to \hom(H_n(C), G) \quad \forall n$

which is surjective.

Proof

Put $Z_n := \ker \partial \subset C_n$, $B_n := \partial(C_{n+1})$.

A class $\alpha \in H^n(C;G)$ is represented by $\varphi: C_n \to G$ s.t. $\varphi \circ \partial = 0$, i.e.

$\varphi|_{B_n} \equiv 0 \Rightarrow \varphi$ descends to

$\bar{\varphi}: \underbrace{Z_n/B_n}_{H_n(C)} \to G$. Define $h(\alpha) := \bar{\varphi}$.

Note that this definition is good, since

If $[\varphi'] = \alpha$, then $\varphi - \varphi' = \Psi \circ \partial$ for some $\Psi : C_{n-1} \to G$. $\Rightarrow \varphi - \varphi' = 0$ on $Z_n \Rightarrow$ $\overline{\varphi'} = \overline{\varphi}$.

Exercise: h is linear.

CLAIM

h is surjective.

Proof

We'll construct a right-inverse to h

$$s : \hom(H_n(C), G) \to H^n(C; G)$$

$$(\ \& \ h \circ s = id)$$

Consider the SES $0 \to Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \to 0$.
B_{n-1} is a subgroup of C_{n-1} and C_{n-1} is free by assumption.
$\Rightarrow B_{n-1}$ is also free. $\Rightarrow$ SES splits.
(submodules of free modules over PIDs are free)

$\Rightarrow \exists\ Z_n \xleftarrow{p} C_n$ a left-inverse of i,

i.e. $p \circ i = id_{Z_n}$. $\Rightarrow \forall$ homo. $\varphi_0 : Z_n \to G$

we can define an extension $\varphi := \varphi_0 \circ p : C_n \to G$

s.t. $\varphi|_{Z_n} = \varphi_0$. The resulting map

$p^* : \hom(Z_n, G) \longrightarrow \hom(C_n, G)$ is

a homomorphism. Now let $\partial \in \hom(H_n(C), G)$.

$\Rightarrow \sigma : Z_n/B_n \to G$. Put

$$\sigma' := (Z_n \to Z_n/B_n \xrightarrow{\sigma} G).$$

Define $\hat{\sigma} := p^*(\sigma') = \sigma' \circ p : C_n \to G$.

We have $\hat{\sigma} \circ \partial = \sigma' \circ p \circ \partial = \sigma' \circ \partial = 0$

$p|_{Z_n} = id$

σ' factors through Z_n/B_n

$\Rightarrow \delta(\hat{\sigma}) = 0$. Define $s(\sigma) := [\hat{\sigma}]$.

Exercise: s is linear. Also $h \circ s(\sigma) = h([\hat{\sigma}]) = \sigma$.

CONCLUSION

h fits into a SES

$$0 \to \ker h \to H^n(C;G) \xrightarrow{h} \hom(H_n(C),G) \to 0$$

which is split (because we've seen $\exists$ a right-inverse s to h)

EXAMPLE

$$C_\bullet = \left(0 \to \underset{\substack{\shortparallel \\ C_3}}{\mathbb{Z}} \xrightarrow{0} \underset{\substack{\shortparallel \\ C_2}}{\mathbb{Z}} \xrightarrow{\times 2} \underset{\substack{\shortparallel \\ C_1}}{\mathbb{Z}} \xrightarrow{0} \underset{\substack{\shortparallel \\ C_0}}{\mathbb{Z}} \to 0 \right)$$

This is the cellular chain complex of $\mathbb{RP}^3$.

$H_0(C_\bullet) = \mathbb{Z}$, $H_1(C_\bullet) = \mathbb{Z}_2$, $H_2(C_\bullet) = 0$, $H_3(C_\bullet) = \mathbb{Z}$.

Take $G = \mathbb{Z}$.

$$C^* = \left(0 \leftarrow \underset{\substack{\shortparallel \\ C_3^*}}{\mathbb{Z}} \leftarrow \underset{\substack{\shortparallel \\ C_2^*}}{\mathbb{Z}} \leftarrow \underset{\substack{\shortparallel \\ C_1^*}}{\mathbb{Z}} \leftarrow \underset{\substack{\shortparallel \\ C_0^*}}{\mathbb{Z}} \leftarrow 0 \right)$$

$H^0(C^*) = \mathbb{Z}$, $H^1(C^*) = 0$, $H^2(C^*) = \mathbb{Z}_2$, $H^3(C^*) = \mathbb{Z}$.

So $h: H^2(C^*) \to \hom(H_2(C), \mathbb{Z})$ has a kernel.

We'll use the following notation

for an abelian group E, write $E^* := \hom(E, G)$.

GOAL: UNDERSTAND BETTER $\ker h$

Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

The rows are SES. Since C_n & C_{n-1} are free, so are B_n & B_{n-1}. $\Rightarrow$ The two rows are split. $\Rightarrow$ after applying $\hom(-, G)$ we get

$$\begin{tikzcd} 0 & Z_{n+1}^* \arrow[l] & C_{n+1}^* \arrow[l, "j^*"'] & B_n^* \arrow[l, "\delta"'] & 0 \arrow[l] \\ 0 & Z_n^* \arrow[l] \arrow[u, "0"] & C_n^* \arrow[l, "j^*"'] \arrow[u, "\delta"] & B_{n-1}^* \arrow[l, "\delta"] \arrow[u, "0"] & 0 \arrow[l] \end{tikzcd}$$

with exact rows.

View $Z^*_\bullet$ & $B^*_\bullet$ as cochain complexes with 0 boundary maps. $\Rightarrow$

$\Rightarrow$ the last diagram is actually a SES of cochain complexes $\Rightarrow$ we get a LES in cohomology

$$\cdots \leftarrow B_n^* \xleftarrow{\tau} Z_n^* \leftarrow H^n(C;G) \leftarrow B_{n-1}^* \xleftarrow{\tau} Z_{n-1}^* \leftarrow \cdots$$

connecting homo.

connecting homo.

CLAIM

τ (the connecting hom) is just i^*, where $i: B_\bullet \to Z_\bullet$ is the inclusion.

In other words τ is just the restriction map.

Use the definition of the boundary homomorphism

Denote by $i_n : B_n \to Z_n$ the inclusion.
Take the previous LES & split it into short exact sequences.

$$0 \leftarrow \ker(i_n^*) \xleftarrow{\tilde{\partial}} H^n(C;G) \leftarrow \operatorname{coker}(i_{n-1}^*) \leftarrow 0$$

Now $\ker(i_n^*) \cong \hom(H_n(C);G)$.

Indeed, if $\varphi : Z_n \to G$ s.t. $\varphi|_{B_n} = 0$, then $\overline{\varphi} : H_n(C) = Z_n/B_n \to G$. And vice versa, if $\overline{\varphi} : H_n(C) \to G$ then we can define $\varphi = (Z_n \to Z_n/B_n \xrightarrow{\overline{\varphi}} G)$ and clearly $i_n^*(\varphi) = 0$.

Denote this iso by

$$\theta : \ker(i_n^*) \to \hom(H_n(C), G).$$

CLAIM

The following diagram commutes

$$\begin{tikzcd} \ker(i_n^*) \arrow[d, "\theta"', "\cong"] & H^n(C;G) \arrow[l, "\tilde{j}"'] \arrow[dl, "h"] \\ \hom(H_n(C),G) & \end{tikzcd}$$

Exercise

We deduce that we have a split SES:

$$0 \to \operatorname{coker}(i_{n-1}^*) \to H^n(C;G) \xrightarrow{h} \hom(H_n(C),G) \to 0$$

Consider the SES:

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0.$$

Dualize it

$$B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \hom(H_{n-1}(C),G) \leftarrow 0.$$

Complete it to an exact sequence.

$$0 \leftarrow \operatorname{coker}(i_{n-1}^*) \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \hom(H_{n-1}(C),G) \leftarrow 0. \quad (*)$$

We'll see soon that $\mathrm{coker}(i_{n-1}^*)$ depends only on $H_{n-1}(C)$ & G.

RESOLUTIONS

Fix an abelian group H. Sometimes we'll view H as a chain complex concentrated at degree 0:

$$\cdots \to 0 \to 0 \to H \to 0 \to 0 \to \cdots$$

We'll denote this chain complex by H.

DEFINITION

A **FREE RESOLUTION** of H is a chain complex $F_\bullet$ with degrees ≥ 0 $(\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \to 0)$ together with a map $F_0 \xrightarrow{\varepsilon} H$ s.t.

1. F_i is free abelian $\forall\, i$
2. The sequence $\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{\varepsilon} H \to 0$ is exact (i.e. the ch. complex has 0 homology)

We'll denote it by $F_\bullet \xrightarrow{\varepsilon} H$.

Exercise: $F_\bullet \to H$ is a free resolution $\Leftrightarrow$

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \to 0$$ & the map

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d] & F_1 \arrow[r, "f_1"] \arrow[d] & F_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[d] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H \arrow[r] & 0
\end{tikzcd}$$

is a chain map that induces an isomorphism in homology (= quasi-isomorphism)

(Replace H with a chain complex whose groups are <u>free</u>).

Given a free resolution of H, apply to it $\hom(-,G)$: we obtain

$$\cdots \leftarrow F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \leftarrow 0$$

Note that

$$\cdots \leftarrow F_2^* \leftarrow \underbrace{F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0}_{\text{exact}}$$, but

the entire cochain complex might not be everywhere exact (acyclic).

Consider the cohomology of the 1st sequence F^*. Denote it by $H^n(F;G)$.

Exercise: $H^0(F;G) \cong H^* = \hom(H,G)$.

$$H^n(F;G) = \ker f_{n+1}^* / \operatorname{Im} f_n^*$$

REMARK

Recall the sequence from the previous discussion

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0$$

this is a free resolution of the group $H := H_{n-1}(C)$.

$(F_0 = Z_{n-1}, F_1 = B_{n-1}, F_i = 0 \ \forall i \geq 2)$

After dualizing we get the following thing

$$\cdots \leftarrow 0 \leftarrow B_{n-1}^* \leftarrow Z_{n-1}^* \leftarrow \hom(H_{n-1}(C), G) \leftarrow 0$$

Note that $\operatorname{coker}(i_{n-1}^*) = H^1(F;G)$.

MAIN LEMMA

1. Let $F_\bullet \to H$ be a free resolution of H and $F_\bullet' \to H'$ a resolution of H' (not necessarily free). Then every homo. $\alpha: H \to H'$ can be extended to a chain map $F_\bullet \to F_\bullet'$, i.e.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "\varepsilon"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r, "f_2'"'] & F_1' \arrow[r, "f_1'"'] & F_0' \arrow[r, "\varepsilon'"'] & H' \arrow[r] & 0
\end{tikzcd}$$

Moreover, any two such extensions are chain homotopic.

2. For every two free resolutions $F_\bullet$ & $F_\bullet'$ of H, $\exists$ canonical isomorphisms

$$H^n(F;G) \cong H^n(F';G) \quad \forall n \geq 0.$$

In other words, $H^n(F;G)$ $n = 0, 1, 2, \ldots$

depend only on H & G (and NOT on the choice of $F_\bullet$).

MAIN POINT

Every abelian group has a free resolution of the type $\dots \to 0 \to F_1 \to F_0 \xrightarrow{\varepsilon} H \to 0$, i.e. with $F_i = 0$ $\forall i \geq 2$. Indeed, choose a set of generators S for H.

Let $F_0 := \bigoplus_{s \in S} \mathbb{Z} \cdot x_s$, where x_s is a symbol corresponding to $s \in S$ (i.e. F_0 is the free abelian group on the set S).

We have a surjective homo $F_0 \xrightarrow{\varepsilon} H$, $\varepsilon(x_s) := s$. Take $F_1 := \ker(\varepsilon_0)$. $F_1 \subset F_0$ is a subgroup $\Rightarrow F_1$ is also free. We get a SES $0 \to F_1 \to F_0 \xrightarrow{\varepsilon} H \to 0$.

CONCLUSION For every free resolution $F_\bullet \to H$ of H we have $H^i(F; G) = 0$ $\forall i \geq 2$. (this follows from the previous lemma + previous short resolution).

So the only two interesting groups are $H^0(F;G)$ & $H^1(F;G)$. We've seen $H^0(F;G) = \hom(H,G)$.

Notation: $\mathrm{Ext}(H,G) := H^1(F;G)$

THEOREM (UNIVERSAL COEFFICIENT THEOREM)

Let $C_\bullet$ be a chain complex of free abelian groups. Let G be an abelian group. Then $\exists$ a split SES:

$$0 \to \mathrm{Ext}(H_{n-1}(C),G) \to H^n(C;G) \xrightarrow{h} \hom(H_n(C);G) \to 0$$

Remark In general $\nexists$ canonical splitting.
'preferred'

How to calculate $\mathrm{Ext}(H,G)$?

PROPOSITION

1. $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$
2. If H is a free abelian group, then

$\mathrm{Ext}(H,G)=0 \quad \forall$ groups G.

③ $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/_{nG} \quad (nG = \{ng : g \in G\} \subset G)$

Remark: the above 3 statements are enough in order to calculate $\mathrm{Ext}(H,G)$ for all finitely generated abelian groups H. This is because we have a SES

$$0 \to \underbrace{H_{torsion}}_{\{h \in H :\ \exists k \in \mathbb{Z} \text{ s.t. } k \cdot h = 0\}} \xrightarrow{inc} H \longrightarrow \underbrace{H/_{H_{torsion}}}_{H_{free}\ -\ \text{free abelian group}} \to 0$$

Since H is finitely generated, H_{free} is a free abelian group, so the sequence splits & we have

$$H \cong H_{free} \oplus H_{torsion} \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{r = \mathrm{rank}(H_{free})} \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}_{k_j} \qquad 2 \leq k_j \in \mathbb{Z},\ \ell \geq 0$$

$$\mathrm{Ext}(H,G) \cong \mathrm{Ext}(H_{tor}, G) \cong \bigoplus_{j=1}^{\ell} \underbrace{\mathrm{Ext}(\mathbb{Z}_{k_j}, G)}_{G/_{k_j G}}$$