THEOREM (UNIVERSAL COEFFICIENT THEOREM) Hatcher

Let $C_\bullet$ be a chain complex of free abelian groups. Let G be an abelian group. Then $\exists$ a split SES:

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} \mathrm{hom}(H_n(C);G) \to 0$$

Remark In general $\nexists$ canonical splitting.
'preferred'

Short review:

① There exists a map
$$h: H^n(C;G) \to \mathrm{hom}(H_n(C), G) \ \forall n$$
which is surjective. ← construct a right inverse s

Put $Z_n := \ker \partial \subset C_n$, $B_n := \partial(C_{n+1})$.

A class $\alpha \in H^n(C;G)$ is represented by $\varphi: C_n \to G$ s.t. $\varphi \circ \partial = 0$, i.e.

$\varphi|_{B_n} \equiv 0$. $\Rightarrow \varphi$ descends to

$$\overline{\varphi} : \underbrace{Z_n / B_n}_{H_n(C)} \to G . \quad \text{Define } h(\alpha) := \overline{\varphi}.$$

(2) h fits into a SES

$$0 \to \ker h \to H^n(C;G) \xrightarrow{h} \hom(H_n(C_\cdot), G) \to 0$$

which is split.

$\ker h$ is not trivial: example $\mathbb{R}P^3$

$h : H^2(C^*) \to \hom(H_2(C), \mathbb{Z})$ has a kernel. $\rightsquigarrow$ we will try to understand this kernel

(3)

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

dualize

$$\begin{tikzcd}
0 & Z_{n+1}^* \arrow[l] & C_{n+1}^* \arrow[l, "j^*"'] & B_n^* \arrow[l, "\delta"'] & 0 \arrow[l] \\
0 & Z_n^* \arrow[l] \arrow[u, "0"] & C_n^* \arrow[l, "j^*"] \arrow[u, "\delta"] & B_{n-1}^* \arrow[l, "\delta"] \arrow[u, "0"] & 0 \arrow[l]
\end{tikzcd}$$

This is a SES of cochain complexes $\leadsto$ it induces a LES of cohomological groups.

connecting hom is just a restriction map i^*

$$\cdots B_n^* \xleftarrow{\tau} Z_n^* \leftarrow H^n(C;G) \leftarrow B_{n-1}^* \xleftarrow{\tau} Z_{n-1}^* \leftarrow \cdots$$

$i_n^* : Z_n^* \to B_n^*$ is induced by

$i_n : B_n \to Z_n$.

We can split the above sequence to

$$0 \to \operatorname{coker}(i_{n-1}^*) \to H^n(C;G) \xrightarrow{\tilde{j}} \ker(i_n^*) \to 0$$

4 $\ker(i_n^*) \cong \hom(H_n(C);G)$ and so we have a SES

$$0 \to \operatorname{coker}(i_{n-1}^*) \to H^n(C;G) \to \hom(H_n(C),G) \to 0$$

The task of finding ker h has been shifted to analyzing $\operatorname{coker}(i_{n-1}^*)$.

It turns out that this mysterious term coker i_{n-1}^* depends only on $H_{n-1}(C)$ and G. ⇝ notation

$$\mathrm{coker}(i_{n-1}^*) = \mathrm{Ext}(H_{n-1}(C), G)$$

(5) UNIVERSAL COEFFICIENT THEOREM FOR COHOMOLOGY

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} \mathrm{hom}(H_n(C);G) \to 0$$

We still owe a more detailed explanation for Ext. To understand it in more depth, we look at free resolutions.

A **FREE RESOLUTION** of H is a chain complex $F_\bullet$ with degrees ≥ 0 $(\dots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \to 0)$ together with a map $F_0 \xrightarrow{\varepsilon} H$ s.t.

① F_i is free abelian $\forall i$

② The sequence $\dots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{\varepsilon} H \to 0$

is exact (ie. the ch. complex has 0 homology)

We'll denote it by $F_\cdot \xrightarrow{\varepsilon} H$.

Apply $\hom(-,G)$ to $F_\cdot$ & get

$$\dots \leftarrow F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \leftarrow 0 .$$

Cohomology groups are $H^n(F;G)$.

MAIN LEMMA: $H^n(F;G)$ depend only on H & G, but <u>not</u> on the choice of $F_\cdot$. (proof coming soon)

[every abelian group has a free resolution of the type $\dots 0 \to F_1 \to F_0 \to H \to 0$

i.e. with $F_i = 0 \quad \forall i \geq 2$

$\Rightarrow$ the only two interesting groups are $H^0(F;G)$ & $H^1(F;G)$.

$H^0(F;G)$ we can compute & see it is $\hom(H,G)$.

$$\mathrm{Ext}(H,G) := H^1(F;G).$$

RETURN TO $\operatorname{coker} i_{n-1}^*$

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0$$

is a free resolution of $H_{n-1}(C)$.

dualize

$$\cdots \leftarrow 0 \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \hom(H_{n-1}(C),G) \leftarrow 0$$

$H_1(F,G) = \operatorname{coker}(i_{n-1}^*)$, where

$(F_0 = Z_{n-1},\ F_1 = B_{n-1},\ F_i = 0 \ \forall i \geq 2)$

$\rightsquigarrow$ From the main lemma it follows that $\operatorname{coker}(i_{n-1}^*)$ depends only on G & $H_{n-1}(C)$.

Exercises:

① Compute Ext groups
$\mathrm{Ext}(\mathbb{Z}_4, \mathbb{Z}_{12})$, $\mathrm{Ext}(\mathbb{Z}_9, \mathbb{Z}_9)$,
$\mathrm{Ext}(\mathbb{Z}_3, \mathbb{R})$, $\mathrm{Ext}(\mathbb{Z}_2, \mathbb{Q}/\mathbb{Z})$

② Suppose that X has integral homology groups

$$H_0(X) = \mathbb{Z}, \quad H_1(X) = \mathbb{Z}_4 \oplus \mathbb{Z}_2,$$
$$H_3(X) = \mathbb{Z}_{72} \oplus \mathbb{Z}$$

and all other groups are 0.
Determine cohomology groups with coefficients in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}_8, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$.

③ Regarding $\mathbb{Z}_2$ as a module over the ring $\mathbb{Z}_4$ construct a resolution of $\mathbb{Z}_2$ by free modules over $\mathbb{Z}_4$ and use this to show that $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all n.

(4) Let X be a topological space. Show that $H^1(X,\mathbb{Z})$ has no torsion.

(5) Compute the singular homology groups with $\mathbb{Z}$ & $\mathbb{Z}_2$ of $\mathbb{R}P^2$ and check the UCT.