PROPOSITION

1. $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$
2. If H is a free abelian group, then

$\mathrm{Ext}(H,G) = 0 \quad \forall$ groups G.

3. $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG \quad (nG = \{ng : g \in G\} \subset G)$

Remark: the above 3 statements are enough in order to calculate $\mathrm{Ext}(H,G)$ for all finitely generated abelian groups H. This is because we have a SES

$$0 \to \underbrace{H_{\text{torsion}}}_{\{h \in H : \exists k \in \mathbb{Z} \text{ s.t. } k \cdot h = 0\}} \xrightarrow{\text{inc}} H \longrightarrow \underbrace{H/H_{\text{torsion}}}_{H_{\text{free}} - \text{ free abelian group}} \to 0$$

PROOF

1. Let $F_\bullet \to H$ be a free resolution of H & $F_\bullet' \to H'$ a free resolution of H'. $\Rightarrow$ $F_\bullet \oplus F_\bullet'$ is a free resolution of $H \oplus H'$.

$$\begin{tikzcd}[row sep=small]
\cdots F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "\varepsilon"] & H \\
\oplus & \oplus & \oplus & \oplus \\
\cdots F_2' \arrow[r, "f_2'"'] & F_1' \arrow[r, "f_1'"'] & F_0' \arrow[r, "\varepsilon'"'] & H'
\end{tikzcd}$$

$(F_i \oplus F_i')^* \cong F_i^* \oplus F_i'^*$

$g_i := f_i \oplus f_i'$

$\Rightarrow g_i^* = f_i^* \oplus f_i'^*$

$$\mathrm{Ext}(H \oplus H', G) = H^1(F_\bullet \oplus F_\bullet'; G) \cong H^1(F_\bullet; G) \oplus H^1(F_\bullet'; G)$$

$$= \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$$

② If H is free, then we can use

$$\begin{tikzcd}
\cdots 0 \arrow[r] & 0 \arrow[r] & H \arrow[r, "id"] & H \arrow[r] & 0 \\
F_2 \arrow[u] & F_1 \arrow[u] & F_0 \arrow[u] & &
\end{tikzcd}$$

as a free resolution of H.

$$\mathrm{Ext}(H, G) = H^1(F; G) = 0.$$

③ Considering the following resolution of $\mathbb{Z}_n$:

$i(a) := na$

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{q} \mathbb{Z}_n \to 0$$

$q :=$ quotient map

Apply $\hom(-,G)$:

hom is determined by image of 1

$$\begin{tikzcd} & F_1^* \arrow[d, phantom, "\shortparallel"] & F_2^* \arrow[d, phantom, "\shortparallel"] \\ 0 & \hom(\mathbb{Z},G) \arrow[l] \arrow[d, phantom, "\cong"] & \hom(\mathbb{Z},G) \arrow[l, "i^*"'] \arrow[d, phantom, "\cong"] \\ & G & G \arrow[l, dashed, "\times n"'] \end{tikzcd}$$

$$\Rightarrow \operatorname{Ext}(\mathbb{Z}_n, G) = H^1(F^*) = G/nG.$$

$\boxtimes$

COROLLARY

Let $C_\bullet$ be a chain complex of free abelian groups. Suppose $H_n(C_\bullet)$ & $H_{n-1}(C_\bullet)$ are finitely generated. Denote by $T_{n-1} \subset H_{n-1}(C_\bullet)$ the torsion subgroup of $H_{n-1}(C_\bullet)$. Then

$$H^n(C;\mathbb{Z}) \cong H_n(C_\bullet)_{\text{free}} \oplus T_{n-1}.$$

The splitting is <u>not</u> canonical.

Exercise: general structure + previous result.

CONCLUSION

Assume that $H_0(C_\bullet)$ is free. Then $H^1(C;\mathbb{Z})$ is free.

The sequence

$$0 \to \mathrm{Ext}(H_{n-1}(C),G) \to H^n(C;G) \to \hom(H_n(C),G) \to 0$$

is natural with respect to chain maps (& homo's of G) in the following sense:

If $\alpha : C_\bullet \to C_\bullet'$ is a chain map, then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}(H_{n-1}(C),G) & \to & H^n(C;G) & \to & \hom(H_n(C),G) & \to 0 \\
& \uparrow \alpha_n^{ext} & & \uparrow \alpha^* & & \uparrow \alpha_n^* & \\
0 \to & \mathrm{Ext}(H_{n-1}(C'),G) & \to & H^n(C';G) & \to & \hom(H_n(C'),G) & \to 0
\end{array}$$

$\alpha_n' : H_n(C) \to H_n(C')$, α_n^* is the dual of α_n'

map induced by α

α^* = the map induced in cohomology from α

α_n^{ext} = map on Ext induced by α_n

Exercise: Prove this.

EXAMPLE FROM TOPOLOGY

$X = \mathbb{RP}^n,\ G = \mathbb{Z}_2$

$0 \to \mathrm{Ext}(H_{i-1}(\mathbb{RP}^n), \mathbb{Z}_2) \to H^i(\mathbb{RP}^n; \mathbb{Z}_2) \to \hom(H_i(\mathbb{RP}^n), \mathbb{Z}_2) \to 0$

Assume $n = \text{even} > 0$. $H_0(X) = \mathbb{Z},\ H_1(X) = \mathbb{Z}_2,$

$H_2(X) = 0, \ldots, H_{2k-1}(X) = \mathbb{Z}_2,\ H_{2k}(X) = 0, \ldots H_n(X) = 0.$

$3 \leq 2k-1 \leq n-1$

Let us calculate

$$\mathrm{Ext}(H_j(X), \mathbb{Z}_2) = \begin{cases} 0 & j = 0 \\ \mathbb{Z}_2 & 1 \leq j = \text{odd} < n \\ 0 & 0 < j = \text{even} \end{cases}$$

$$\hom(H_i(X), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & i = 0 \\ \mathbb{Z}_2 & 1 \leq i = \text{odd} < n \\ 0 & 0 < i = \text{even} \end{cases}$$

$$\Rightarrow H^i(X; \mathbb{Z}_2) \cong \mathrm{Ext}(H_{i-1}(X), \mathbb{Z}_2) \oplus \hom(H_i(X), \mathbb{Z}_2)$$
$$\cong \mathbb{Z}_2 \quad \forall\ 0 \leq i \leq n$$

Exercise: Carry out the calculation of

$H^i(\mathbb{RP}^n; \mathbb{Z}_2)$ for n = odd.

MAIN LEMMA

① Let $F_\bullet \to H$ be a free resolution of H and $F_\bullet' \to H'$ a resolution of H' (not necessarily free). Then every homo. $\alpha : H \to H'$ can be extended to a chain map $F_\bullet \to F_\bullet'$, i.e.

$$
\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "\varepsilon"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r, "f_2'"'] & F_1' \arrow[r, "f_1'"'] & F_0' \arrow[r, "\varepsilon'"'] & H' \arrow[r] & 0
\end{tikzcd}
$$

Moreover, any two such extensions are chain homotopic.

② For every two free resolutions $F_\bullet$ & $F_\bullet'$ of H, $\exists$ canonical isomorphisms

$$H^n(F; G) \cong H^n(F'; G) \quad \forall n \geq 0.$$

In other words, $H^n(F; G)$ $n = 0, 1, 2, \ldots$

depend only on H & G (and <u>NOT</u> on the choice of $F_\cdot$).

Proof of the lemma

$f_0 := \varepsilon$

$f_0' := \varepsilon'$

We will define α_i by induction. $\alpha_{-1} := \alpha$

F_0 is free, so we choose a basis $\{x_s\}$.

$f_0' = \varepsilon'$ is surjective, so $\exists\, x_s' \in F_0'$ s.t. $f_0'(x_s') = \alpha f_0(x_s)$. Define $\alpha_0(x_s) := x_s'$.

Since F_0 is free this defines α_0 uniquely & $f_0'\alpha_0 = \alpha_{-1} f_0$. Now suppose we've already defined $\alpha_{-1}, \alpha_0, \dots, \alpha_i$.

$$\begin{tikzcd}
F_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, dashed, "\alpha_{i+1}"'] & F_i \arrow[r, "f_i"] \arrow[d, "\alpha_i"] & F_{i-1} \arrow[r, "f_{i-1}"] \arrow[d, "\alpha_{i-1}"] & \cdots \arrow[r] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha_{-1}"] & 0 \\
F_{i+1}' \arrow[r, "f_{i+1}'"'] & F_i' \arrow[r, "f_i'"'] & F_{i-1}' \arrow[r, "f_{i-1}'"'] & \cdots \arrow[r] & F_0' \arrow[r, "f_0'"'] & H \arrow[r] & 0
\end{tikzcd}$$

Again, we choose a basis $\{x_s\}$ for F_{i+1}.

For every basis element x_s,

$\alpha_i f_{i+1}(x_s) \in \operatorname{Im}(f'_{i+1})$ because

$\alpha_i f_{i+1}(x_s) \in \ker(f'_i)$ ($\leftarrow$ this is because

$f'_i \alpha_i f_{i+1}(x_s) = \alpha_{i-1} f_i f_{i+1}(x_s) = 0$).

Define $\alpha_{i+1}(x_s) := x_s'$, where

$x_s' \in (f'_{i+1})^{-1}(\alpha_i f_{i+1}(x_s))$.

This proves the existence of $\alpha_j \ \forall j$.

<u>Uniqueness up to chain homotopy</u>

If $\{\alpha_i\}$ & $\{\alpha_i'\}$ are two extensions of α, we have to show that $\alpha_i - \alpha_i'$ is chain homotopic to 0. Note that $\{\alpha_i - \alpha_i'\}$ is an extension of $0: H \to H'$.

So it's enough to prove that if $\{\beta_i\}$ is an extension of $H \xrightarrow{0} H'$

then $\exists$ a chain homotopy $h_i : F_i \to F'_{i+1}$, $i = -1, 0, \ldots$

s.t. $f'_{i+1}\circ h_i + h_{i-1}\circ f_i = \beta_i$

$$\begin{tikzcd}
\cdots \arrow[r] & F_1 \arrow[r] \arrow[d, "\beta_1"] \arrow[dl, "h_0"'] & F_0 \arrow[r, "f_0"] \arrow[d, "\beta_0"] \arrow[dl, "h_{-1}"'] & H \arrow[r] \arrow[d, "0"] & 0 \\
\cdots \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r, "f_0'"'] & H \arrow[r] & 0
\end{tikzcd}$$

Induction on i:

For $i=-1$, take $h_{-1}=0$. Then h_0 has to satisfy $\beta_0(x)=f_1' h_0(x)$ $\forall x\in F_0$.

Again, choose a basis $\{x_s\}$ for F_0.

$\forall$ basis element x_s, $\beta_0(x_s)\in \ker(f_0')=\mathrm{Im}(f_1')$.

So, $\exists$ $x_s'\in F_1'$ s.t. $f_1'(x_s')=\beta_0(x_s)$.

Define $h_0(x_s):=x_s'$.

Let $i\geq 1$. Suppose we've already defined $h_{-1}, h_0, \dots, h_{i-1}$ s.t. $f_i' h_{i-1}+h_{i-2}f_{i-1}=\beta_{i-1}$.

We'll now define h_i, s.t.

$$\begin{tikzcd}
\arrow[r] & F_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\beta_{i+1}"] & F_i \arrow[r, "f_i"] \arrow[d, "\beta_i"] \arrow[dl, "h_i"'] & F_{i-1} \arrow[r, "f_{i-1}"] \arrow[d, "\beta_{i-1}"] \arrow[dl, "h_{i-1}"'] & F_{i-1} \arrow[d, "\beta_{i-2}"] \arrow[dl, "h_{i-2}"'] \\
\arrow[r] & F_{i+1} \arrow[r, "f_{i+1}'"'] & F_i \arrow[r, "f_i'"'] & F_{i-1} \arrow[r, "f_{i-1}'"'] & F_{i-1}
\end{tikzcd}$$

$$f_{i+1}' h_i(x) + h_{i-1} f_i(x) = \beta_i(x) \quad \forall x \in F_i.$$

Choose a basis $\{x_s\}$ of F_i. If we know that $y_s := \beta_i(x_s) - h_{i-1} f_i(x_s) \in \mathrm{Im}(f_{i+1}') = \ker(f_i')$

then we are done: just define $h_i(x_s) := x_s'$ for some choice $x_s' \in F_{i+1}'$ s.t. $f_i'(x_s') = y_s$. Indeed,

$$f_i'(y_s) = f_i' \beta_i(x_s) - f_i' h_{i-1} f_i(x_s) =$$

$$\underset{\text{I.P.}}{=} \beta_{i-1} f_i(x_s) - (\beta_{i-1} - h_{i-2} f_{i-1}) \circ f_i(x_s)$$

$$= \beta_{i-1} f_i(x_s) - \beta_{i-1} \circ f_i(x_s) + \cancelto{0}{h_{i-2} \circ f_{i-1} \circ f_i(x_s)}$$

$$= 0$$

$\Rightarrow y_s \in \ker(f_i')$ as we wished. This completes the induction. (this proves part 1)

Let G be an abelian group & suppose we have two extensions $\{d_i\}, \{\gamma_i\}$

of $\alpha : H \to H'$. Consider $\alpha^* : H'^* \to H^*$ and the cochain maps $\alpha_i^* : F_i'^* \to F_i^*$, $\beta_i^* : F_i'^* \to F_i^*$.

$$\begin{array}{ccccc} 0 \to & H^* & \to & F_0^* & \to \cdots \\ & \alpha^* \uparrow & & \alpha_0^* \uparrow\uparrow \beta_0^* & \\ 0 \to & H'^* & \to & F_0'^* & \to \cdots \end{array}$$

Since $\alpha_\cdot \simeq \beta_\cdot \Rightarrow \alpha_\cdot^* \simeq \beta_\cdot^*$

chain homotopies — cochain homotopies

$\Rightarrow$ the induced maps in cohomology coincide: $\alpha_i^* = \beta_i^* : H^i(F'; G) \to H^i(F; G)$.

In particular, we get a canonical map

$$\alpha^{ext} : \mathrm{Ext}(H', G) \to \mathrm{Ext}(H, G)$$

that depends only on $\alpha : H \to H'$.

Now, let H, H', H'' be abelian groups & $F_\bullet, F'_\bullet, F''_\bullet$ resolutions of H, H', H'' respectively with $F_\bullet$ & $F'_\bullet$ being free.

Let $H \xrightarrow{\alpha} H' \xrightarrow{\beta} H''$ be a homo.

$$\Rightarrow (\beta \circ \alpha)_i^* = \alpha_i^* \circ \beta_i^* : H^i(F''_\bullet ; G) \to H^i(F_\bullet ; G) \quad (*)$$

In particular,

$$(\beta \circ \alpha)^{ext} = \alpha^{ext} \circ \beta^{ext} : Ext(H'', G) \to Ext(H, G)$$

The reason for this $(*)$ is that we can choose the extension of $\beta \circ \alpha$ to be $\beta_i \circ \alpha_i : F_i \to F''_i \ \forall i$.

$$\begin{tikzcd}
F_\bullet \arrow[r, "f"] \arrow[d, "\alpha_\bullet"'] \arrow[dd, bend right=60, "\beta_\bullet \circ \alpha_\bullet"'] & H \arrow[d, "\alpha"] \\
F'_\bullet \arrow[r, "f'"'] \arrow[d, "\beta_\bullet"'] & H' \arrow[d, "\beta"] \\
F''_\bullet \arrow[r, "f''"'] & H''
\end{tikzcd}$$

Issue: there are many possible resolution - why is Ext well defined?

Consider now two free resolutions $F_\bullet$ & $F_\bullet'$ of the same group H. We want to show that there is a canonical isomorphism $H^1(F;G) \cong H^1(F';G)$, hence $\mathrm{Ext}(H,G)$ is well-defined (i.e. independent of the free resolution of H, up to iso).

Let's consider $id: H \to H$. We obtain two possible extensions of this map.

$\alpha_\bullet : F_\bullet \to F_\bullet'$ and $\beta_\bullet : F_\bullet' \to F_\bullet$.

$$\begin{tikzcd}
F_\bullet \arrow[r, "f"] \arrow[d, dashed, "\alpha"'] & H \arrow[d, "id"] \\
F_\bullet' \arrow[r, "f'"] \arrow[d, dashed, "\beta"'] & H \arrow[d, "id"] \\
F_\bullet \arrow[r, "f"'] & H
\end{tikzcd}$$

Now

$\beta_\bullet \circ \alpha_\bullet : F_\bullet \to F_\bullet$ is an extension of id. Also, $id_\bullet : F_\bullet \to F_\bullet$ is an extension of identity.

$\alpha_i^* \circ \beta_i^* = (\beta_i \circ \alpha_i)^* = id^* = id : H^i(F;G) \to H^i(F;G)$

Similarly,

$$\beta_i^* \circ \alpha_i^* = (\alpha_i \circ \beta_i)^* = id^* = id : H^i(F;G) \to H^i(F;G)$$

$\Rightarrow \alpha_i^*$ & β_i^* are isomorphisms.

Moreover, α_i^* & β_i^* are canonical.

In particular, $(\beta \circ \alpha)^{ext} = id^{ext} = id$

$$\begin{gathered} (\beta \circ \alpha)^{ext} \\ \parallel \\ \alpha^{ext} \circ \beta^{ext} \end{gathered}$$

UCT FOR TENSOR PRODUCTS

Recall: Let $U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be an exact sequence of R-modules. Then $\forall$ R-mod M, the sequence

$$M \otimes_R U \xrightarrow{id \otimes f} M \otimes_R V \xrightarrow{id \otimes g} M \otimes_R W \to 0$$

is exact. But $\exists$ SES $0 \to U \to V \to W \to 0$ for which we lose exactness from the left $M \otimes_R -$.

If M is free $\Rightarrow 0 \to M \otimes_R U \to M \otimes_R V \to M \otimes_R W \to 0$ is exact. Also, if $0 \to U \to V \to W \to 0$ splits $\Rightarrow 0 \to M \otimes_R U \to M \otimes_R V \to M \otimes_R W \to 0$ is exact $\forall$ R-mod M.

Let $(C_\bullet, \partial)$ be a chain complex of free abelian groups. ($R = \mathbb{Z}$, $\otimes = \otimes_{\mathbb{Z}}$)

Question: What is the relation between $H_*(C_\bullet \otimes G)$ and $H_*(C_\bullet), G$?

diff. is $\partial \otimes id$

we will denote this also by $H_*(C_\bullet; G)$

Like before, consider $B_k \subset Z_k \subset C_k$, denote by $i_k : B_k \to Z_k$ the inclusion. Consider the commutative diagram below. It is a SES of chain complexes, where on $B_\bullet$ & $Z_\bullet$ we take the differential 0.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"'] & C_n \arrow[r, "\partial"] \arrow[d, "\partial"'] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$B_K \subset C_K$ is free because C_K is free $\Rightarrow$ every row of the sequence is split. (not necessarily the chain complex).

If we tensor with G, we still obtain SESs:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \otimes G \arrow[r] \arrow[d, "0"] & C_n \otimes G \arrow[r] \arrow[d, "d"] & B_{n-1} \otimes G \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes G \arrow[r] \arrow[d] & C_{n-1} \otimes G \arrow[r] \arrow[d] & B_{n-2} \otimes G \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$d_i = \partial \otimes id$

Passing to homology we get a LES

$$\cdots \to B_n \otimes G \xrightarrow{c_n} Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{c_{n-1}} \cdots$$

conn. hom.

CLAIM:

$c_n = i_n \otimes id$, $c_{n-1} = i_{n-1} \otimes id$ (exercise)

(Follows from the definition of the connecting homomorphism)

Now break the LES into many SESs:

$$0 \to \operatorname{coker}(i_n \otimes id) \to H_n(C;G) \to \ker(i_{n-1} \otimes id) \to 0 \quad (*)$$

LEMMA

Let $f: U \to V$ be a homomorphism of R-modules. Let M be an R-module. Consider $f \otimes id : U \otimes_R M \to V \otimes_R M$. Then $\exists$ a canonical iso

$$\operatorname{coker}(f \otimes id) \cong \operatorname{coker}(f) \otimes_R M.$$

Proof

Let N be an R-modul & $I \subset N$