Passing to homology we get a LES

$$\cdots \to B_n \otimes G \xrightarrow{c_n} Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{c_{n-1}} \cdots$$

conn. hom.

CLAIM:

$c_n = i_n \otimes id$, $c_{n-1} = i_{n-1} \otimes id$ (exercise)

(Follows from the definition of the connecting homomorphism)

Now break the LES into many SESs:

$$0 \to \operatorname{coker}(i_n \otimes id) \to H_n(C;G) \to \ker(i_{n-1} \otimes id) \to 0 \qquad (*)$$

LEMMA

Let $f: U \to V$ be a homomorphism of R-modules. Let M be an R-module. Consider $f \otimes id: U \otimes_R M \to V \otimes_R M$. Then $\exists$ a canonical iso

$$\operatorname{coker}(f \otimes id) \cong \operatorname{coker}(f) \otimes_R M.$$

Proof

Let N be an R-modul & $I \subset N$

a submodule. Consider

$$0 \to I \xrightarrow{i} N \xrightarrow{p} N/I \to 0$$

This is a SES of R-modules. Let M be an R-module. $\Rightarrow$

$$I \otimes_R M \xrightarrow{i \otimes id} N \otimes_R M \xrightarrow{p \otimes id} (N/I) \otimes_R M \to 0$$ is

exact. $\Rightarrow$ $\dfrac{N \otimes M}{\operatorname{Im}(i \otimes id)} \cong (N/I) \otimes_R M$

(homework)

Apply this to $N = V$, $I = \operatorname{Im}(f)$. So $N/I = \operatorname{coker}(f)$. Also note that

$$\operatorname{im}(i \otimes id) = (i \otimes id)(f(U) \otimes M) = $$
$$= \operatorname{im}(f \otimes id)$$

$\boxtimes$

Going back to $\circledast$ and applying the lemma we have

$$\operatorname{coker}(i_n \otimes \mathrm{id}) \cong \operatorname{coker}(i_n) \otimes G = H_n(C) \otimes G.$$

$$\Rightarrow 0 \to H_n(C)\otimes G \to H_n(C;G) \to \ker(i_{n-1}\otimes \mathrm{id}) \to 0$$

$a = [c]$

$g \in G$

$$\underbrace{\underset{\substack{\nearrow \\ \text{generators of this group}}}{a \otimes g} \longmapsto [c \otimes g]}_{\text{exercise: check that this holds}}$$

Analyzing $\ker(i_k \otimes \mathrm{id})$

Consider free resolutions $F_\bullet \xrightarrow{\varepsilon} H$ of a given abelian group H. Given another abelian group G, we consider $F_\bullet \otimes G \to 0$, which is a chain complex. $\rightsquigarrow$

$$H_i(F_\bullet; G) := H_i(F_\bullet \otimes G).$$

THEOREM

$\forall$ two free resolutions $F_\bullet \xrightarrow{\varepsilon} H$, $F_\bullet' \xrightarrow{\varepsilon'} H$,

$\exists$ a canonical iso $H_n(F_\bullet;G) \cong H_n(F_\bullet';G)$.

So, $H_*(F;G)$ depends only on H & G up to canonical iso.

Exercise 1: $H_0(F;G) \cong H \otimes G$

Exercise 2: $H_i(F;G) = 0 \quad \forall i \geq 2$

take a very short res.

TOR FUNCTOR

Define

$$\mathrm{Tor}(H,G) := H_1(F_\bullet;G).$$

If $\alpha : H \to H'$ is a homo $\Rightarrow$ $\exists$ a canonical map

$$\alpha_{tor} : \mathrm{Tor}(H,G) \to \mathrm{Tor}(H',G).$$

And $H \xrightarrow{\alpha} H' \xrightarrow{\beta} H'' \rightsquigarrow (\beta \circ \alpha)_{tor} = \beta_{tor} \circ \alpha_{tor}$

Consider now $H := H_{n-1}(C)$. We have a SES

$$0 \to \underset{\deg 1}{B_{n-1}} \xrightarrow{i_{n-1}} \underset{\deg 0}{Z_{n-1}} \to H_{n-1}(C) \to 0$$

which gives a free resolution of $H_{n-1}(C)$.

After $-\otimes G$, and taking H_1 we get $\ker(i_{n-1} \otimes id) = \mathrm{Tor}(H_{n-1}(C), G)$.

THEOREM

Let $(C_\cdot, \partial)$ be a chain complex of free abelian groups, and G and abelian group. Then $\exists$ a SES

$$0 \to H_n(C) \otimes G \to H_n(C_\cdot \otimes G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0.$$

This sequence is natural with respect to chain maps $C_\cdot \to C'_\cdot$ as well as w.r.t. homo. $G \to G'$. Moreover, the sequence splits (but not canonically).

Proof that SES of homological UCT splits.

$$0 \to H_n(C_\cdot) \otimes G \to H_n(C_\cdot \otimes G) \to \mathrm{Tor}(H_{n-1}(C_\cdot), G) \to 0$$

Consider the sequence

$$0 \to Z_n \xrightarrow{j_n} C_n \xrightarrow{\partial} B_{n-1} \to 0$$

This sequence splits because B_{n-1} is free (b.c. C_{n-1} is free).

$\Rightarrow \exists\ Z_n \xleftarrow{p} C_n$ which is a left inverse of j_n, ie. $p|_{Z_n} = id$. Compose p with the quotient map $Z_n \to H_n(C_\cdot)$, and we get $\bar{p}: C_n \to H_n(C_\cdot)$ and we have $\bar{p}(z) = [z]\ \forall z \in Z_n$.

This map exists $\forall n$, so we can view it as a chain map $\bar{p}: C_\cdot \to H_*(C_\cdot)$, where $H_*(C_\cdot)$ is viewed as a chain complex with 0-differential (indeed $\bar{p}(\partial c) = [\partial c] = 0$). Tensoring with G we get a chain map

$$\bar{p} \otimes id : C_\cdot \otimes G \to H_*(C_\cdot) \otimes G.$$

Passing to homology we get

$$(\bar{p} \otimes id)_* : H_n(C_\cdot \otimes G) \to H_n(C_\cdot) \otimes G.$$

CLAIM

$(\bar{p}\otimes id)_*$ is a left inverse of

$$H_n(C_\bullet)\otimes G \xrightarrow{h} H_n(C_\bullet\otimes G).$$

Proof of claim

Let $a=[c]\in H_n(C_\bullet)$, with $c\in Z_n$,

and let $g\in G$. $\Rightarrow h(a\otimes g)=[c\otimes g]$

$\Rightarrow (\bar{p}\otimes id)_*([c\otimes g])=\bar{p}(c)\otimes g=$

$$=[c]\otimes g=a\otimes g. \Rightarrow$$

$$(\bar{p}\otimes id)_*\circ h=id.$$

$\blacksquare$

PROPERTIES OF TOR

① $Tor(A,B)\cong Tor(B,A)$

(Tor is a symmetric functor)

② If A or B are free, then $Tor(A,B)=0$

③ $Tor(\bigoplus_{i\in I}A_i,B)\cong\bigoplus_{i\in I}Tor(A_i,B).$

④ Let A be finitely generated and let

$A_{torsion} \subset A$ be the torsion subgroup of A.
Then $Tor(A,B) \cong Tor(A_{torsion}, B)$.

⑤ $Tor(\mathbb{Z}_m, A) \cong \ker(A \xrightarrow{\times m} A)$.

①-⑤ $\Rightarrow$ calculation of $Tor(A,B)$
for all finitely generated abelian groups
A & B.

Example $Tor(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_k$, where

$k = \gcd(n,m)$.

Proof: Consider the free resolution of $\mathbb{Z}_n$:

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}_n \to 0.$$

After $\otimes \mathbb{Z}_m$ we get

$$0 \to \mathbb{Z}_m \to \mathbb{Z}_m \to 0 \quad \text{for}$$

the non-augmented chain complex.

$$\mathrm{Tor}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \ker(\mathbb{Z}_m \xrightarrow{\times n} \mathbb{Z}_m)$$

Exercise $\cong \mathbb{Z}_k \quad k := \gcd(n,m)$.

Property 5 is proved in a very similar way. Note that also

$$\mathbb{Z}_k \cong \mathbb{Z}_n \otimes \mathbb{Z}_m$$

Moreover, the isos in the properties section are canonical. The isos 1,2,4 are natural w.r. to homomorphisms of groups for any two of the factors in $\mathrm{Tor}(-,-)$.

The iso in 5 is natural w.r. to homomorphisms of groups for the 2nd factor i.e. $A' \xrightarrow{\alpha} A''$ gives

$$\begin{tikzcd}
\mathrm{Tor}(\mathbb{Z}_m, A') \arrow[r, "\cong"] \arrow[d, "\alpha_{tor}"'] \arrow[dr, phantom, "\mathrm{C}"] & \ker(A' \xrightarrow{\times m} A') \arrow[d, "\text{induced by } \alpha"] \\
\mathrm{Tor}(\mathbb{Z}_m, A'') \arrow[r, "\cong"'] & \ker(A'' \xrightarrow{\times m} A'')
\end{tikzcd}$$

COROLLARY

If A, B are finitely generated abelian groups $\Rightarrow \mathrm{Tor}(A,B) \cong A_{\text{torsion}} \otimes B_{\text{torsion}}$

Proof

$$A \cong A_{\text{free}} \oplus A_{\text{torsion}}$$

$$B \cong B_{\text{free}} \oplus B_{\text{torsion}}$$

$$\Rightarrow A_{\text{torsion}} \cong \bigoplus_{i=1}^{r} \mathbb{Z}_{n_i}$$

$$B_{\text{torsion}} \cong \bigoplus_{j=1}^{\ell} \mathbb{Z}_{m_j}$$

this requires assumptions that both A & B are finitely generated

Outline of the proofs

③ Assume A is free. We have a very short free resolution of A:

$$\cdots \to 0 \to A \xrightarrow{\mathrm{id}} A \to 0$$

deg 1   deg 0   deg -1

$\Rightarrow \mathrm{Tor}(A,B) = 0 \quad \forall B.$

Assume B = free. Pick a free abelian group F which surjects onto A

$$F \xrightarrow{\text{surj.}} A$$

Let $R \subset F$ be the kernel of that surjection. R is also free abelian. $\rightsquigarrow$

we get a free resolution of A

$$\dots \to 0 \to R \to F \to A \to 0$$

Since B is free, $- \otimes B$ keeps the sequence exact, so the sequence

$$0 \to \underset{\text{deg } 1}{R \otimes B} \to \underset{\text{deg } 0}{F \otimes B} \to A \otimes B \to 0$$

is exact, $\Rightarrow$ Tor$(A, B) = 0$.

(2) Take a free resolution of each i & their direct sums. Take a tensor product & get direct sum of tensor products.

④ $A = A_{free} \oplus A_{Torsion}$ + use property ③

⑤ Take the following free resolution of $\mathbb{Z}_m$:

$$\cdots \to 0 \to \underset{\text{deg } 1}{\mathbb{Z}} \xrightarrow{\times m} \underset{\text{deg } 0}{\mathbb{Z}} \to \mathbb{Z}_m \to 0$$

After $- \otimes A$: $\quad \ldots\ 0 \to A \xrightarrow{\times m} A \to \mathbb{Z}_m \otimes A \to 0$

$\Rightarrow \operatorname{Tor}(\mathbb{Z}_m, A) = \ker(A \xrightarrow{\times m} A)$

(homology in degree 1)

$\blacksquare$

To conclude the topic:

EXT & TOR for other rings & modules

It is not always true that a submodule of a free R-module is free, for general rings. Free resolutions don't