Oral Exam Preparation

Here is a list of sample questions for the oral exam. In addition to this, you should also carefully review the homework problems and examples that we did in class together.

1. State the definition of the tensor product of R-modules $\mathscr{U}$ and $\mathscr{V}$.
2. Show that for any two R-modules $\mathscr{U}$ and $\mathscr{V}$, a tensor product of $\mathscr{U}$ and $\mathscr{V}$ exists.
3. Show that $\mathscr{U} \otimes \mathscr{V} \cong \mathscr{V} \otimes \mathscr{U}$ for R-modules $\mathscr{U}$ and $\mathscr{V}$.
4. Let X be a topological space and G an abelian group. Define homology groups $\mathrm{H}_i(X; G)$ of X with coefficients in G via a chain complex. What are reduced homology groups with coefficients in G?
5. Compute homology groups of an n-sphere S^n with coefficients in G, where G is an abelian group.
6. How are homology groups with different coefficients related (hint: long exact sequence) and what is the Bockstein homomorphism?
7. Let $f \colon S^n \to S^n$ be a continuous map. How is the degree of f defined if we take homology groups with coefficients in $\mathbb{Z}$? How about if we take coefficients in G?
8. Define cellular homology groups of a CW complex with coefficients in an abelian group G.
9. Compute cellular homology groups of $\mathbb{R}P^n$ with coefficients in an abelian group G.
10. State and prove the Borsuk-Ulam Theorem. What is the Smith exact sequence or the Gysin sequence?
11. State and prove the Lusternik-Schnirelmann Theorem using the Borsuk-Ulam Theorem.
12. State and prove the Ham Sandwich Theorem using the Borsuk-Ulam Theorem.
13. Define what a co-chain complex is. What is the Hom functor? What are singular cohomology groups with coefficients in G?
14. What is a free resolution of an abelian group H? Given another abelian group G, how is $\mathrm{Ext}(H, G)$ defined? List some properties of $\mathrm{Ext}(H, G)$.
15. State the Universal Coefficient Theorem for cohomology and describe the main steps in the proof.
16. State the Universal Coefficient Theorem for homology with coefficients and describe the main steps in the proof. Given abelian groups H and G, how is $\mathrm{Tor}(H, G)$ defined?
17. What is a graded abelian group? What is the tensor product of two graded abelian groups? How about graded chain complexes?

18. State the Eilenberg-Zilber Theorem. How are the isomorphisms in the theorem defined? What method of proof is used to prove this theorem?

19. State the algebraic Künneth formula and give a sketch of the proof.

20. State the topological version of the Künneth formula and use it to compute homology groups of the n-torus.

21. Define the cup product. How does the boundary of the cup product relate to cup products of boundaries?

22. Show that cup product is well defined on the level of cohomology groups. Is it commutative? What is the cohomology ring?

23. Describe the cohomology rings of closed surfaces.

24. What is a manifold? What is the local homology of a manifold M at $x \in M$? What about the local orientation? What does it mean for a manifold to be closed?

25. What is the orientation 2-sheeted cover $\tilde{M}$? How many connected components can $\tilde{M}$ have? What information does the number of connected components of $\tilde{M}$ give about M? Provide proofs for these statements.

26. How is $\tilde{M}_{\mathbb{Z}}$ defined? What is an R-orientation on M (state the definition involving $\tilde{M}_{\mathbb{Z}}$)?

27. What is the fundamental class of a manifold M and when does it exist? Provide a sketch of the proof.

28. Define the cap product. How does the boundary of the cap product relate to cap products of boundaries? In what sense is the cap product natural? Show some examples of cap product calculations for closed surfaces.

29. How is cohomology with compact supports defined? Give an interpretation in terms of limits of cohomology groups of pairs. Compute $\mathrm{H}_c^*(\mathbb{R}^n; G)$.

30. State the Poincare Duality Theorem. Sketch the proof.