Musterlösung Serie 11

LAVER FORCING UND MATHIAS FORCING

32. Given $\mathbb{L}_{\mathscr{F}} = (L, \leqslant)$, for each $f \in {}^{\omega}\omega \cap \mathbf{V}$, consider the set of conditions given by

$$D_f = \{T \in L : \exists\, n \in \omega\, \forall\, k > n\, \forall\, t \in T[k]\, (next_T(t) \cap (f(k)+1) = \varnothing)\},$$

D_f is clearly open, and we want to argue that it is dense as well. Fix some $\tilde{T} \in L$. Let $s \in {}^{<\omega}\omega$ be the stem of $\tilde{T}$ and write $n = \mathrm{dom}(s)$. Consider now the subtree of $\tilde{T}$ given by

$$T = \{t \in \tilde{T} : t \preceq s \vee (s \preceq t \wedge (\forall\, k \in \mathrm{dom}(t) \setminus n,\, t(k) > f(k)))\}.$$

Since by assumption $\mathscr{F}$ includes the Fréchet filter, we have that $T \in L$. To see that, notice that we know by definition of $\mathbb{L}_{\mathscr{F}}$ that for all nodes $t \succeq s$ in $\tilde{T}$ we have $next_{\tilde{T}}(t) \in \mathscr{F}$. Moreover, $\mathscr{F}$ is a filter and for all $k \in \omega$ we have $\omega \setminus k \in \mathscr{F}$, from which we deduce that $next_{\tilde{T}}(t) \cap (\omega \setminus k) = next_{\tilde{T}}(t) \setminus k$ belongs to $\mathscr{F}$ as well (it is in particular infinite). Hence, $T \in L$ finally follows from the fact that s is the stem of T and that for all nodes $t \succeq s$ in T we have $next_{\tilde{T}}(t) \setminus k = next_T(t)$. We can now conclude since clearly $T \geqslant \tilde{T}$ and if $g \in {}^{\omega}\omega$ is the Laver real given by the intersection of the conditions in the generic filter, then T forces that for all $k > n$, $g(k) > f(k)$, which, by the fact that f was arbitrary, shows that g is a dominating real.

33. In what follows, for each subset $s \in \mathscr{P}(\omega)$ let $\tilde{s}$ be the only increasing function $\tilde{s}\colon |s| \to \omega$ such that $\mathrm{Im}(\tilde{s}) = s$. Moreover, if for some $\alpha \in \omega + 1$ we have two functions $f, g\colon \alpha \to \omega$, define $\gamma_{f,g}\colon \alpha \to 2$ by $\gamma_{f,g}(k) = 1$ if and only if $f(k) < g(k)$. We provide an argument that works both for Laver and for Mathias Forcing, since it relies on the fact that conditions can be written in form (s, t_s) for some stem $s \in {}^{<\omega}\omega$ and some infinite set t_s (respectively a perfect tree and a subset of ω) which restricts how s can grow in stronger conditions. If G is a generic filter for $\mathbb{M} \times \mathbb{M}$, then G is in the form $G_1 \times G_2$, where each G_i is a generic filter for $\mathbb{M}$. Call m_1 and m_2 the Mathias reals $\in [\omega]^{\omega}$ corresponding to G_1 and G_2. We claim that $\gamma_{\tilde{m_1}, \tilde{m_2}}$ is a Cohen real. Indeed, consider the subset of $L \times L$ given by

$$E = \{\langle (s, x_s), (t, x_t)\rangle \in L \times L : |s| = |t|\},$$

together with the embedding

$$\Gamma\colon E \to \bigcup_{n \in \omega} {}^{n}2, \text{ defined by } \Gamma(\langle (s, x_s), (t, x_t)\rangle) = \gamma_{\tilde{s}, \tilde{t}}.$$

Notice that E is a dense subset of $L \times L$. Now, if D is an open dense set in $\bigcup_{n \in \omega} {}^{n}2$ and e is some condition in E, a moment's thought shows that there is a stronger condition

$e' \geqslant e$ with $\Gamma(e') \in D$, which means that $\Gamma^{-1}(D)$ is dense in E, and is consequently dense in $L \times L$. We can now deduce that $G_1 \times G_2$ intersects $\Gamma^{-1}(D)$, and hence, since D was arbitrary, the filter given by the initial segments of $\gamma_{\tilde{m}_1,\tilde{m}_2}$ is generic for $\bigcup_{n\in\omega} {}^n2$, which is what we wanted to show.

34. Assume that $\mathscr{U}$ is not a Ramsey ultrafilter, while $r \in {}^\omega\omega$ is $\mathbb{M}_\mathscr{U}$-generic. Let $\pi\colon [\omega]^2 \to 2$ with $\pi \in \mathbf{V}$ be a colouring such that there is no homogeneous $x \in \mathscr{U}$ with respect to π. We now want to show that for all $n \in \omega$ the set

$$D_n = \{(s, x_s) \in M : (s, x_s) \Vdash_{\mathbb{M}_\mathscr{U}} |\mathrm{Im}(r) \cap \pi^{-1}(0)| \geqslant n \wedge |\mathrm{Im}(r) \cap \pi^{-1}(1)| \geqslant n\}$$

is open dense in $\mathbb{M}_\mathscr{U}$. As usual, D_n is clearly open. In order to show that it is also dense, fix $n \in \omega$ and let $(t, x_t) \in M$ be an arbitrary but fixed condition. Notice that it suffices to show that there exists some $k \in \omega$ such that $|x_t \cap k \cap \pi^{-1}(0)| \geqslant n$ and $|x_t \cap k \cap \pi^{-1}(1)| \geqslant n$, for if we are able to find such an initial segment then we can consider the condition $(t \cup (x_t \cap k), x_t \setminus k)$, which is stronger than (t, x_t) and which belongs to D_n, as we remind that the $\mathbb{M}_\mathscr{U}$-generic real $r \in {}^\omega\omega$ is obtained as the unique strictly increasing function from ω to ω that satisfies $\mathrm{Im}(r) = \bigcup_{(s,x_s)\in G} s$, where G is the $\mathbb{M}_\mathscr{U}$-generic filter. The existence of such an initial segment follows by the fact that since there is no π-homogeneous set in $\mathscr{U}$, then no element of $\mathscr{U}$ can be almost-homogeneous for π. More explicitely: assume towards a contradiction that for all $k \in \omega$ we have (we pick wlog the *colour* 1) that $|x_t \cap k \cap \pi^{-1}(1)| < n$. Then there is a $k_0 \in \omega$ such that for all $k \geqslant k_0$ we have $|x_t \cap k \cap \pi^{-1}(1)| = |x_t \cap k_0 \cap \pi^{-1}(1)|$, which means that no elements in $x_t \setminus k_0$ can be involved in a pair P belonging to $[x_t]^2$ such that $\pi(P) = 1$. Now, since $\mathscr{U}$ is a non-principal ultrafilter and $x_t \in \mathscr{U}$, we get that also $(x_t \setminus k_0) \in \mathscr{U}$, but $x_t \setminus k_0$ is π-homogeneous, contradicting the choice of π. To summarize, we showed that for every element $x \in \mathscr{U}$ and for every $n \in \omega$ we can find an initial segment of x which is sufficiently *non-homogeneous* with respect to the appositely chosen π, and this allows us to conclude that D_n is dense in $\mathbb{M}_\mathscr{U}$. Notice that by the fact that n was arbitrary, the real number r is not almost-homogeneous for π. On the other hand, by Exercise 2 (see Serie 1), we know that if a real number $r' \in {}^\omega\omega$ is $\mathbb{L}_\mathscr{U}$-generic, then the set $\mathrm{Im}(r') \subset \omega$ is almost homogeneous for every colouring $\pi\colon [\omega]^2 \to 2$ with $\pi \in \mathbf{V}$. We can thus conclude that for r to be both $\mathbb{L}_\mathscr{U}$- and $\mathbb{M}_\mathscr{U}$-generic, it is necessary that $\mathscr{U}$ is a Ramsey ultrafilter.