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ON THE CONSISTENCY OF $\omega_1 = \mathfrak{s} = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$

35. This is Proposition 18.3 in [1], to which we refer for the details. We solve simultaneously (a) and (b) by showing that for every non-zero countable ordinal γ we have $\mathbb{C} \approx \mathbb{C}\gamma$ and that for every ordinal λ we have the three equivalences $\mathbb{C}^\lambda \approx \mathbb{C}_\lambda$, $\mathbb{C}^\lambda \approx \mathbb{C}^{|\lambda|}$ and $\mathbb{C}_\lambda \approx \mathbb{C}_{|\lambda|}$. It would suffice to find a dense embedding for each of these pairs, but we'll actually describe isomorphisms between the corresponding partially ordered sets.

$\mathbb{C} \approx \mathbb{C}\gamma$: Given that γ is countable, we can fix a bijection between $\omega \times \gamma$ and ω, which induces an isomorphism between the possible domains in $\mathbb{C}$ and $\mathbb{C}\gamma$.

$\mathbb{C}^\lambda \approx \mathbb{C}_\lambda$: Since $\mathbb{C}^\lambda$ is defined as a finite support product, each $\mathbb{C}^\lambda$-condition p has finite domain, so it can be bijectively mapped to an element of $\mathrm{Fn}(\omega \times \lambda, 2)$.

$\mathbb{C}_\lambda \approx \mathbb{C}_{|\lambda|}$: Fix a bijection between λ and $|\lambda|$. This induces an isomorphism between the finite subsets of $\omega \times \lambda$ and $\omega \times |\lambda|$, as well as an order-isomorphism between $\mathbb{C}^\lambda$-conditions and $\mathbb{C}^{|\lambda|}$-conditions, which finishes the proof.

36. This is Lemma 22.12 in [1]. Let f be a function in ${}^\omega\omega^{\mathbf{V}[G]}$, and let $\underset{\sim}{f}$ be a name such that there is a $\mathbb{C}^{\omega_1}$-condition $\tilde{p} \in G$ with $\tilde{p} \Vdash \underset{\sim}{f} \in {}^\omega\omega$. Fixed the name $\underset{\sim}{f}$, we can define in the ground model for each $\mathbb{C}^{\omega_1}$-condition $p \geqslant \tilde{p}$ a function in ${}^\omega\omega$ as follows:

$$f_p(n) = \min\{k \in \omega : \exists\, q \geqslant p\, (q \Vdash \underset{\sim}{f}(n) = k)\}.$$

We'd like to remark that we haven't so far used any particular property of the forcing notion $\mathbb{C}^{\omega_1}$. Consider the family $\mathscr{F} = \{f_p : p \geqslant \tilde{p} \text{ is a } \mathbb{C}^{\omega_1}\text{-condition}\}$. Since $|\mathscr{F}| = \omega_1$ and by assumption $\omega_1 < \mathfrak{b}^{\mathbf{V}}$, we can find in the ground model a function $g_f \in {}^\omega\omega$ which dominates $\mathscr{F}$. We get by choice of g_f and construction of each f_p that for all $\mathbb{C}^{\omega_1}$-conditions $p \geqslant \tilde{p}$, we have that if for some $h \in {}^\omega\omega^{\mathbf{V}}$, $p \Vdash h <^* f$, then $h <^* f_p$, and consequently $h <^* g_f$. This means that, in order to dominate every function, a family in the generic extension must be at least as large as a dominating family in the ground model, finishing the proof.

37. This is Proposition 22.13 in [1]. Following the hint, start with a model $\mathbf{V} \models \mathsf{ZFC} + \mathfrak{p} = \omega_1 < \mathfrak{c}$, and force with $\mathbb{C}^{\omega_1}$, obtaining a family $\mathscr{C} = \{c_\alpha : \alpha \in \omega_1\}$ of Cohen reals. More strongly: each c_γ is a Cohen real over $\mathbf{V}[\langle c_\alpha : \alpha \in \gamma\rangle]$, see Lemma 22.9 in [1] for a proof. Since $\mathbb{C}^{\omega_1}$ satisfies *ccc*, we know that $\omega_1^{\mathbf{V}[G]} = \omega_1^{\mathbf{V}} < \mathfrak{c}^{\mathbf{V}} \leqslant \mathfrak{c}^{\mathbf{V}[G]}$. Moreover, we know that at stage ω_1 we do no add any new real, that is, every real belonging to $\mathbf{V}[G]$ appears at some countable stage (see Lemma 18.9 in [1]). We can then deduce that given any $f \in {}^\omega\omega^{\mathbf{V}[G]}$, say appearing at stage $\gamma \in \omega_1$, f can not be dominating $c_{\gamma+1}$, as

Cohen reals are unbounded. This shows that $\mathscr{C}$ is unbounded and hence $\mathfrak{b} \leqslant \omega_1$, which clearly implies $\mathfrak{b} = \omega_1$. Repeating the same argument together with Lemma 22.3 in [1] yields $\mathfrak{s} = \omega_1$. On the other hand, we know that we didn't add *too many* new reals and hence $\mathfrak{c}^{\mathbf{V}} = \mathfrak{c}^{\mathbf{V}[G]}$, so we can conclude by Exercise 36 that $\mathfrak{d}^{\mathbf{V}} = \mathfrak{d}^{\mathbf{V}[G]}$, concluding the proof.

Literatur

[1] Lorenz Halbeisen, **Combinatorial Set Theory: With a Gentle Introduction to Forcing**, (revised and extended second edition), [Springer Monographs in Mathematics], Springer, London, 2017.