Musterlösung Serie 5

Properties of Forcing Extensions

12. We show a stronger statement, namely that if $\mathbb{P}$ is a κ-closed partial ordering and $f\colon \theta \to X$ is a function in the corresponding generic extension from some cardinal $\theta \in \kappa$ to some set $X \in \mathbf{V}$ already existing in the ground model, then $f \in \mathbf{V}$ as well.

    Let G be a generic filter on $\mathbb{P} = (P, \leqslant)$ and let $f\colon \theta \to X$ be a function with $f \in \mathbf{V}[G]$, together with a $\mathbb{P}$-name $\underset{\sim}{f}$ for f. By the properties of the Forcing Relation (Thm. 15.10 (2)) we can find a condition $p \in G$ such that

    $$p \Vdash \underset{\sim}{f} \in {}^{\theta}\underset{\sim}{X},$$

    where ${}^{\theta}\underset{\sim}{X}$ is a name for the set of all functions from θ to X in $\mathbf{V}[G]$. We will proceed by induction, with the base step given by the fact that, according to Lemma 15.11 (b), there is a condition $p_0 \geqslant p$, $p_0 \in G$, which decides the image of 0 through f, that is, more formally, there is a condition $p_0 \geqslant p$, $p_0 \in G$, and an element $x_0 \in X$ such that $p_0 \Vdash \underset{\sim}{f}(\dot{0}) = \dot{x}_0$. We now define an analogous condition p_λ for all $\lambda \in \theta$. If $\lambda = \lambda' + 1$ is a successor ordinal, then repeat the base step in order to obtain a $p_\lambda \geqslant p_{\lambda'}$, $p_\lambda \in G$, and some $x_\lambda \in X$ with $p_\lambda \Vdash \underset{\sim}{f}(\dot{\lambda}) = \dot{x}_\lambda$. Let now $\lambda \in \theta$ be a limit ordinal. We now want to argue that we can find a condition $\tilde{p_\lambda} \in G$ such that for all $\mu \in \lambda$ we have that $\tilde{p_\lambda} \geqslant p_\mu$. Consider the set

    $$D_\lambda = \{p \in P : \forall\, \mu \in \lambda\ (p \geqslant p_\mu) \text{ or } \exists\, \mu \in \lambda\ (p \perp p_\mu)\}.$$

    D_λ is clearly open. In order to show that it is also dense, let $g \in P$ be an arbitrary condition not in D_λ such that for all conditions $h \geqslant g$, there is no $\mu \in \lambda$ with $h \perp p_\mu$. We are then able to find a condition $g_0 \in P$ with $g \leqslant g_0 \geqslant p_0$. For each successor ordinal $\mu \in \lambda$, $\mu = \mu' + 1$, define inductively g_μ with $g_{\mu'} \leqslant g_\mu \geqslant p_\mu$. For limit ordinals $\mu \in \lambda$, first find by κ-closedness some condition $\tilde{g_\mu}$ with $\tilde{g_\mu} \geqslant g_\alpha$ for all $\alpha \in \mu$, and then, since $\tilde{g_\mu} \geqslant g$ and $\mu \in \lambda$, we are by assumption able to find a g_μ with $\tilde{g_\mu} \leqslant g_\mu \geqslant p_\mu$. Finally, an element of D_λ above g is given by any upper bound of the sequence $\langle g_\mu : \mu \in \lambda\rangle$, whose existence is again guaranteed by $\mathbb{P}$ being κ-closed, concluding the proof that D_λ is an open dense subset of P. Consider now an element in the non-empty intersection $x \in G \cap D_\lambda$. Since G is directed we necessarily have that x is compatible with p_μ for every $\mu \in \lambda$, which, by definition of D_λ, implies that $x \geqslant p_\mu$ for all $\mu \in \lambda$, which is what we wanted for our condition $\tilde{p_\lambda}$. Now apply again Lemma 15.11 (b) and get a $p_\lambda \in G$ with $p_\lambda \Vdash \underset{\sim}{f}(\dot{\lambda}) = \dot{x}_\lambda$. Let now $q \in P$ be any upper bound for the sequence $\langle p_\mu : \mu \in \theta\rangle$.

We get that q decides the image through f of every element of θ, and hence, by definability of forcing, we get that we are able to define f in the ground model, which shows that $f \in \mathbf{V}$.

13. Let $\mathbb{P} = (P, \leqslant)$ be a partially ordered set satisfying the κ-chain-condition (in short κ-cc) for some regular cardinal κ, and let G be a generic filter on $\mathbb{P}$. Let λ and θ be cardinals in V satisfying $\lambda < \theta$ and $\kappa \leqslant \theta$. Let now $f\colon \lambda \to \theta$ be a function belonging to the generic extension $\mathbf{V}[G]$. The claim will follow by showing that f can not be surjective. Let $\underset{\sim}{f}$ be a name for f and $p \in P$ a condition such that

$$p \Vdash \underset{\sim}{f} \in {}^{\theta}\underset{\sim}{\lambda},$$

where ${}^{\theta}\underset{\sim}{\lambda}$ is a name for the set of all functions from θ to λ in $\mathbf{V}[G]$. Consider now for each $\alpha \in \lambda$ the set of conditions above p which decide the image of α through f, formally

$$D_\alpha = \{q \geqslant p : \exists\, \gamma \in \theta\ (q \Vdash \underset{\sim}{f}(\underset{\cdot}{\alpha}) = \underset{\cdot}{\gamma}\}.$$

We'd like to show that every D_α is dense above p. In order to do that, fix $\alpha \in \lambda$ and consider, for each $\gamma \in \theta$, the set $\Delta_{\alpha,\gamma}$ of conditions which decide the forcing sentence $\underset{\sim}{f}(\underset{\cdot}{\alpha}) = \underset{\cdot}{\gamma}$. By Fact 15.9, each $\Delta_{\alpha,\gamma}$ is open dense in P. Assume now towards a contradiction that there is some $q \geqslant p$ such that for all $q' \geqslant q$ we have that $q' \notin D_\alpha$. By Definition 15.8 (c) of the forcing relation, we get that for all $\gamma \in \theta$, q is such that

$$q \Vdash \underset{\sim}{f}(\underset{\cdot}{\alpha}) \neq \underset{\cdot}{\gamma}. \tag{1}$$

On the other hand, since $q \geqslant p$, we also have $q \Vdash \underset{\sim}{f} \in {}^{\theta}\underset{\sim}{\lambda}$, which implies $p \Vdash \underset{\sim}{f}(\underset{\cdot}{\alpha}) \in \underset{\cdot}{\theta}$, contradicting (1), since we can assume that there is a generic filter $\tilde{G}$ on $\mathbb{P}$ containing q (see Chapter 16). Let us now define for each $\alpha \in \lambda$ the set

$$Y_\alpha = \{\gamma \in \theta : \exists\, q \in D_\alpha\ (q \Vdash \underset{\sim}{f}(\underset{\cdot}{\alpha}) = \underset{\cdot}{\gamma})\},$$

which we remark belonging to the ground model $\mathbf{V}$. Notice now that $|Y_\alpha| < \kappa$ by the fact that $\mathbb{P}$ satisfies the κ-cc. Indeed, let μ and δ be distinct elements of Y_α, and let q_μ and q_δ be some respectively corresponding conditions. Clearly $q_\mu \perp q_\delta$, which proves the upper bound on $|Y_\alpha|$. Define now the union

$$Y = \bigcup_{\alpha \in \lambda} Y_\alpha.$$

If $\kappa < \theta$ then clearly $|Y| < \theta$, as $|Y| \leqslant \lambda \cdot \kappa = \max(\lambda, \kappa) < \theta$. If $\kappa = \theta$, we get again $|Y| < \theta$, for regularity of κ, together with $\lambda < \kappa$, implies that there is some cardinal $\mu < \theta$ such that for all $\alpha \in \lambda$ we have $|Y_\alpha| \leqslant \mu$, from which we deduce $|Y| \leqslant \lambda \cdot \mu = \max(\lambda, \mu) < \theta$. We conclude by noticing that $Y \subsetneq \theta$ together with

$$\forall\, \alpha \in \lambda\ (p \Vdash \underset{\sim}{f}(\underset{\cdot}{\alpha}) \in \underset{\cdot}{Y})$$

implies that f can not be surjective, which means that θ is not collapsed in the generic extension.