Musterlösung Serie 6

Models of finite fragments of ZFC

14. The proof goes exactly as in exercise 16 (a). Indeed, we remark that, except for the fact that it is not uncountable, ω satisfies all the requirements for being an inaccessible cardinal:

    - $\aleph_0$ is a limit cardinal.
    - $\aleph_0$ is regular.
    - For all cardinals λ, we have that $\lambda < \aleph_0$ implies $2^\lambda < \aleph_0$.

    This allows us to use exactly the same arguments we'll use to show that V_κ is a model of ZFC for κ inaccessible. To avoid confusion, we remark that V_ω does not satisfy the Axiom of Infinity.

15. *Weak solution:* With the same arguments as in exercise 16 (a), one easily shows that V_{ω_1}, equipped with the standard $\in$-relation, satisfies the four required axioms.

    Strong solution: We can actually construct a countable set-model for the four reuired axioms. Let us start considering $M_0 = \omega + 1 = \omega \cup \{\omega\}$. Our aim is to extend M_0 to some set M, and we do that according to the following definition by induction. Assume that for some $n \in \omega$, M_n is already defined. Then

    $$M_{n+1} = M_n \cup \{\{x, y\} : x \in M_n \wedge y \in M_n \wedge x \neq y\},$$

    and finally $M = \bigcup_{i\in\omega} M_i$. Now, since $\omega + 1 \subseteq M$, clearly M satisfies the Axiom of Empty Set and the Axiom of Infinity. Moreover, M is transitive, as $\omega + 1$ is transitive and, for $m > 0$, every element added to build M_m is a subset of M_{m-1}, which implies that transitivity is preserved. We conclude that M is extensional as every transitive subset of $\mathbf{V}$ is extensional (again, see exercise 16 (a), Axiom of Extensionality). To conclude, let x and y be two distinct elements of M. Let n_x and n_y be the least natural numbers such that $x \in M_{n_x}$ and $y \in M_{n_y}$, and assume wlog that $n_x \geqslant n_y$. By construction $\{x, y\} \in M_{n_x+1} \subseteq M$, which shows that $(M, \in)$ satisfies the Axiom of Pairing.